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Acceleration of Machine Parts

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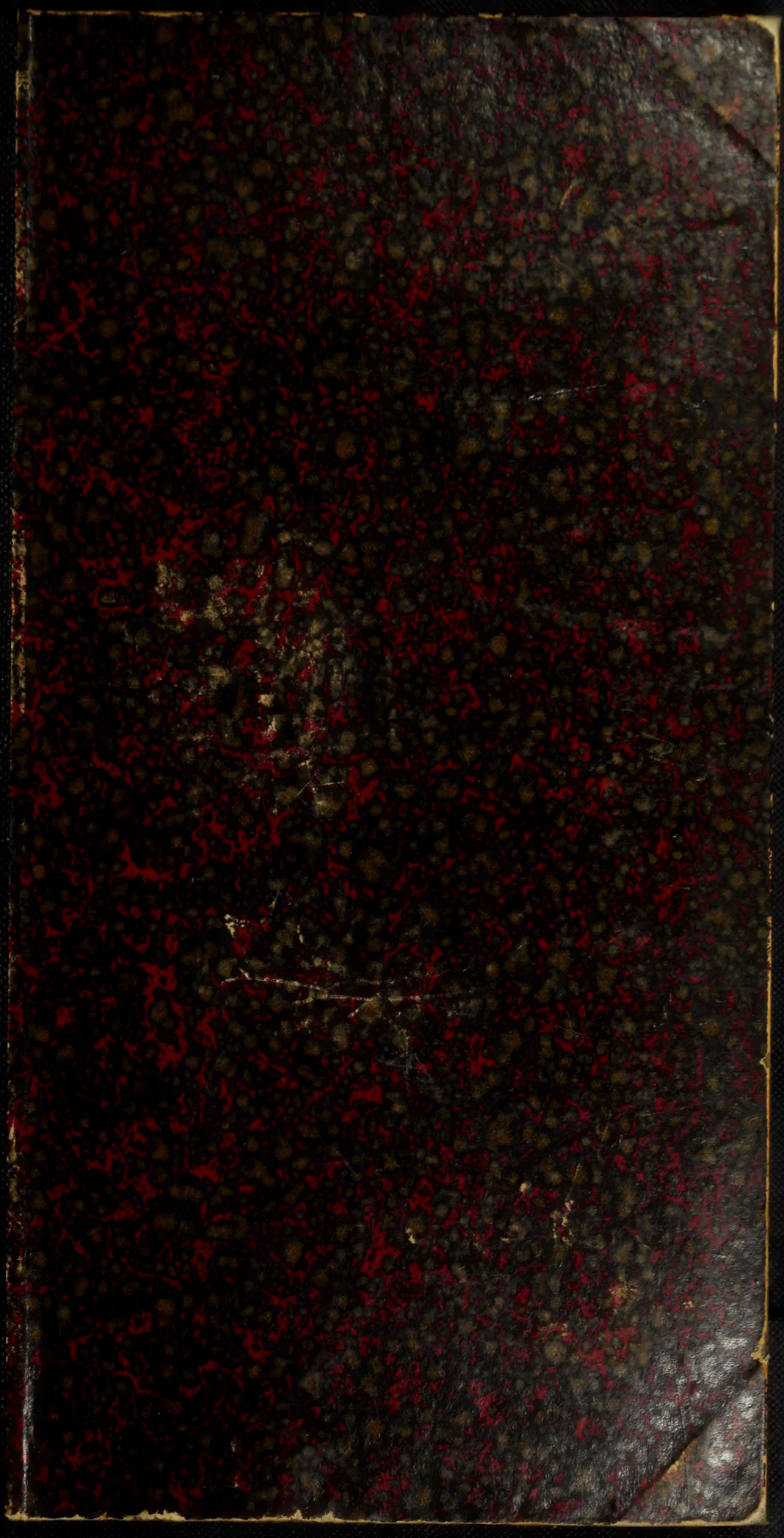

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THE ACCELERATION OF MACHINE PARTS

BY

WILLIAM VAN DUNKIN

THESIS FOR THE DEGREE OF BACHELOR OF SCIENCE
IN MECHANICAL ENGINEERING

IN THE
COLLEGE OF ENGINEERING
OF THE
UNIVERSITY OF ILLINOIS
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THIS IS TO CERTIFY THAT THE THESIS PREPARED UNDER MY SUPERVISION BY

WILLIAM VAN DUNKIN

ENTITLED THE ACCELERATION OF MACHINE PARTS

IS APPROVED BY ME AS FULFILLING THIS PART OF THE REQUIREMENTS FOR THE DEGREE

OF Bachelor of Science in Mechanical Engineering.

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HEAD OF DEPARTMENT OF Mechanical Engineering.

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The Acceleration of Machine Parts

Introductory:— Mechanics deals with matter, space, and time in their measurable relations. The investigation of the measurable relations of space alone is called geometry. If to the idea of space, that of time be added; both in their measurable relations there results the idea of motion. Here arises a separate branch of physical science to which the name kinematics is given. We may define kinematics as the study of pure motion alone. In kinematics, therefore, we are not concerned with the cause of motion, nor the configuration of the moving system. The study of displacements, velocities, and accelerations of geometrical magnitudes is the subject matter of the science. Endowing these geometrical magnitudes with mass, we

are led to the ideas of momentum, force, energy, etc. In the study of force and energy the idea of time may or may not enter. This branch of mechanics, called dynamics, is divided into two parts, statics and kinetics. The former is independent of time, the latter is not. In the former, the forces are in equilibrium, in the latter the forces may or may not be in equilibrum.

Kinematics then forms the introduction to mechanics. In this thesis the ideas of motion, displacement, speed, velocity, and acceleration will be rereived briefly. After this, the purpose is to enter into a fuller discussion of the subject of acceleration in the rigid system.

Vector Quantites. — For convenience, quantities may be separated into two kinds: the one kind called Scaler quantities, the other Vector quantities. Any quantity which is completely defined by the simple statement of its magnitude is called a Scaler quantity. Thus, one bushel, ten foot pounds, ten cubic feet are all examples of this kind of quantity. Quantites of this kind are added and subtracted by the ordinary rules of arithmetic. There are

quantities, however, in which the simple statement of their magnitudes is not enough to define them. Thus, displacement, velocity or acceleration of a point, force, couples, a current of water are all examples of such quantities. These are called Vector quantities. They involve the idea of direction as well as magnitude. In general a Vector quantity is said to have—

Magnitude.
Direction.
Sense.

A Vector quantity may be represented by a line, whose length is proportional to the magnitude of the quantity, and whose direction is parallel to the direction of the quantity; the sense being indicated by an arrow placed upon the line. Thus in Fig. 1 the length M N represents, say a force 10 pounds, whose direction is parallel to M N and whose sense is from M to N. The line is referred to as the vector MN. If the lower line represents a + quantity, a reversal of sign, denoted by a reversal of the arrow-head, shown on the upper line of Fig. 1 is

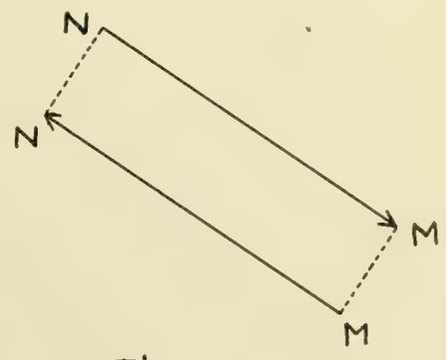


Fig. 1.

another vector quantity spoken of as $-MN$ or $+NM$. Thus a change of sign, or a reversal of the letters designating a line, is the same thing as reversing the sense of the vector.

The rule for the addition of vectors is stated as follows:- Starting from any point, lay off the lines representing the vector quantities, as if to form a polygon, being sure to have the arrow-heads pointing round in the same direction; the line drawn from the starting-point closing the polygon, represents their sum.

Thus let a point, Fig. 2, be acted upon by five forces represented in magnitude and direction by the vectors ab, bc, cd, de and ef, and intersecting in the point O. From any point A lay off AB equal and parallel to vector ab. In the same

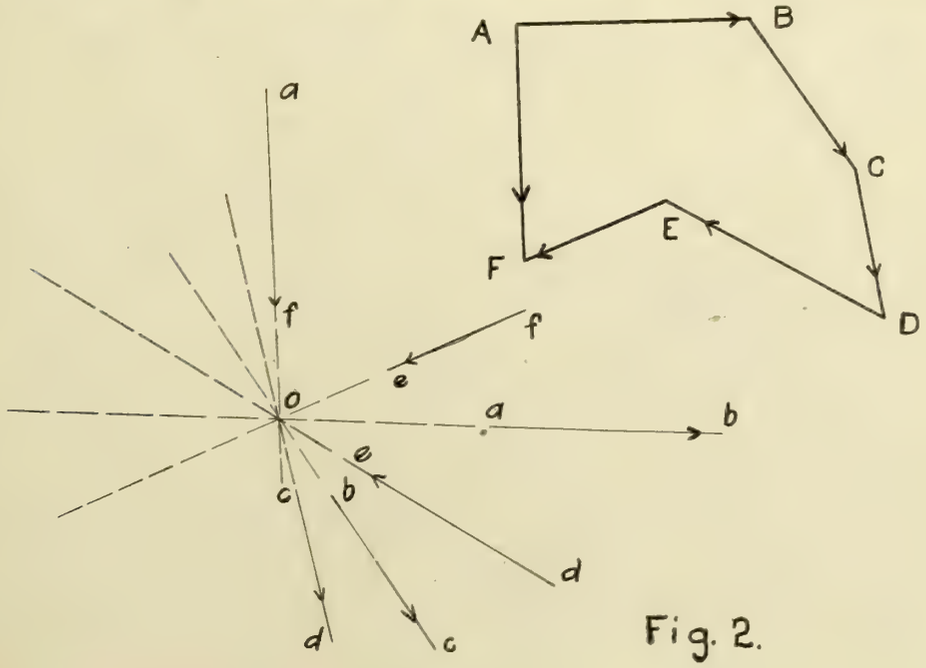


Fig. 2.

way lay off BC, CD and DE. The polygon thus formed is ABCDEF, and the sum or resultant of the five forces is the closing side AF.

The subtraction of two vectors is performed by setting them out from the same point: the lines joining their ends represents their difference. Thus if we have two vectors AB and BC, their vector sum is AB + BC which is equal to AC or

$$AB + BC = AC$$

By transposing we may write the preceding equation

$$AB - AC = -BC = +CB$$

The corresponding triangle is shown in Fig. 3. The rule for placing the arrow-head on the resultant is, that it must always be placed in circuit with the quantity to be subtracted.

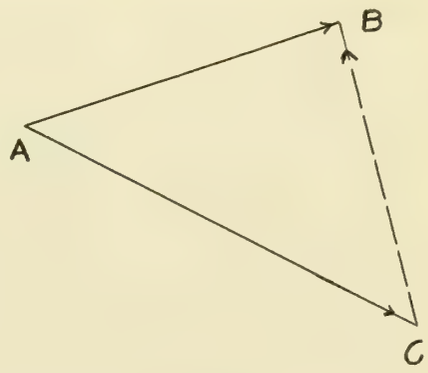


Fig. 3.

Part I

Motion.— A point moves when the straight lines joining it to the point of reference change either in length, or inclination to each other. Hence, motion is a change in position. The line through the successive positions of a point is called its path. It may be any kind of a line whatsoever. If the path is without angles or abrupt changes of direction, it is called a continuous path. Three kinds of motion may be referred to. First, a motion of translation; second, a motion of rotation and third, a combined motion of translation and rotation. The direction of motion at any point is the geometrical tangent to the path or curve at that point. Hence, the direction changes from point to point along the curve. The total change of direction of a curve is called its total curvature; and is measured by the angle formed by the tangents at the extremities of the curve. The total curvature divided by the length is the mean curvature The actual curvature at any point of a curve is the limit approached by the mean curva-

ture of the arc, beginning at the point in question, as the length of the arc is indefinitely decreased.

Speed.— Time rate of motion without reference to direction is called mean speed. It may be variable or uniform. It is variable when the mean speed varies with the interval of time. A point moving with uniform speed must pass over equal distances in equal time units. The limiting value of the mean speed when the interval of time is indefinitely small is the instantaneous speed. If the instantaneous speed at any instant is equal to the mean speed for any interval of time, the speed is uniform. When the instantaneous speed is variable, it has different values for the equal intervals of time. Let OA, Fig. 4 be any distance, measured along the path of a moving point from a fixed point O, taken as origin. Also let OB measured along the path denote the final distance from the same origin. Calling these distances S_1 and S_2 respectfully and t the units of time occupied in describing the distance AB we have,

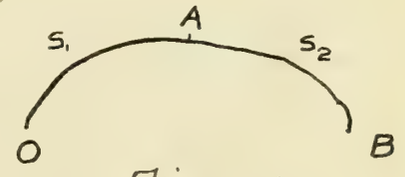


Fig. 4

$$(1)\quad \text{Mean Speed} \equiv V = \frac{AB}{t} = \frac{S_2 - S_1}{t}$$

(2) Instantaneous Speed $\equiv v = \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$.

From (1) and (2), speed has both magnitude and sign but is independent of direction.

(Angular Speed — Let a point move in any path whatsoever from any initial position A to any final position B. [Fig. 5] Choose any point O as a pole, and let the radii vectors OA and OB be drawn. Then the angle θ is called the angular revolution of the point A referred to O. The straight line AB is the linear displacement of the point. θ is measured in radians and is, therefore independent of the length of the radii vectors. The angular revolution of a moving point about an axis will be measured by the angle included between the perpendiculars let fall from the initial and final positions of the point to the axis. Mean angular speed is the angular revolution per unit of time. The limiting value of the mean angular speed when the interval of time is indefinitely decreased is called the instantaneous angular speed. If θ_1 is the angle measured in the plane of revolution from any fixed line to the initial position of the radius vector and θ_2 to the final position of the radius vector, then the

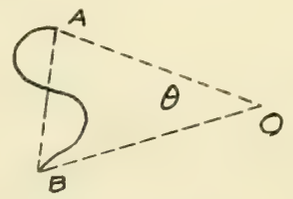


Fig. 4.

expression for mean angular speed is

$$\omega = \frac{\theta_2 - \theta_1}{t}.$$

Suppose two positions of the radii : vectors very near each other: then in the increment notation we shall have

$$(3) \quad \omega = \frac{\Delta\theta}{\Delta t}$$

which on passing to the limit becomes

$$(4) \quad \omega = \lim_{\Delta t \doteq 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}.$$

These equations are the same in form as those for mean and instantaneous speed and will become identical by replacing θ in (3) and (4) with s.

Rate of Change of Speed.— When the speed of a point is not constant the difference between the final and initial instantaneous speeds for any interval of time is called the total change of speed. The total change of speed per unit of time is the mean rate of change of speed. Mean rate of change of speed may be variable or uniform. When the units which give the mean rate of change of speed are variable in number for equal time units, the mean speed is variable. If these units are the same for equal units of time, the mean speed is uniform. The limiting values of the mean rate of change of speed, when the time is indefinitely decreased, is called the instantaneous rate of change of speed.

Rate of change of speed may be zero, uniform or variable. If the rate of change of speed is zero, the speed is uniform and is equal to the mean speed for the interval of time. When the rate of change is uniform it has the same value as the mean rate of change for any interval of time. When change of speed is variable, the mean rate has different values for the same intervals of time.

Let V and V_1 denote speeds at two points, not coincident, in the path of motion. Let t be the time during which the speed changed from V to V_1; then for mean rate of change of speed $\equiv \alpha$

$$(5) \quad \alpha = \frac{V - V_1}{t}$$

If the points approach very near each other, we shall have

$$(6) \quad \alpha = \frac{\Delta V}{\Delta t}$$

which, upon passing to the limit becomes

$$(7) \quad \alpha = \lim_{\Delta t = 0} \frac{\Delta V}{\Delta t} = \frac{dV}{dt} = \frac{d^2 S}{dt^2}.$$

From the preceding equations, it can be seen that if V_1 is greater than V, then the value of α will be negative. Hence we say that motion is retarded. When α is positive, then velocity is increasing. These signs merely show that V is either decreasing or increasing, numerically. Nothing is indicated concerning direction. Rate of change of speed,

therefore, can not be represented by a straight line.

Displacement.— If a point moves from A to B the distance AB is the linear displacement of the point no matter what the path of motion may have been from A to B. The displacement is completely represented by the straight line AB. The length of the line gives the magnitude of the displacement and the direction as indicated by an arrow gives the direction of the displacement. Displacement has both magnitude and direction; and is, therefore, a vector quantity.

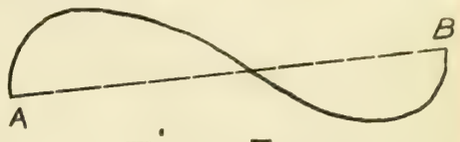


Fig. 5.

Polygon of Displacements.— Since displacement of a point is change of position, it can only be determined by reference to a chosen point of reference. Let AB, BC, and CD be the given displacements of A relative to B, B relative to C, and C relative to D. If AB and BC be laid off in the given direction and magnitude, the line AC gives the displacement of A relative to C. Similarly, AD gives the displacement relative to D. Hence, if any

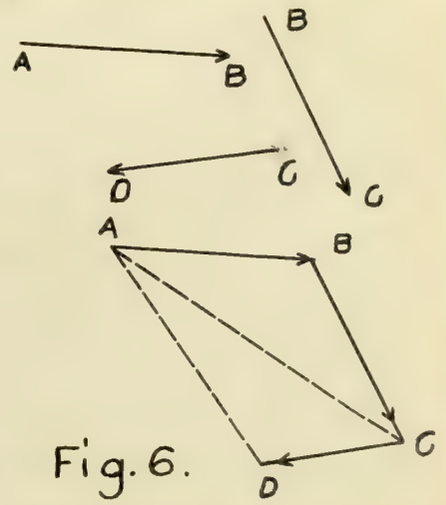


Fig. 6.

number of displacements in the same plane are taken in order and a polygon formed, the line which closes the polygon taken the opposite way around will give the magnitude and direction of the resultant displacement of the first point relative to the last, and taken the same way around, of the last point relative to the first. This principle holds true whether the displacements are successive or simultaneous.

By an application of the above principle any number of given displacements may be resolved into two or more components in any given direction. Thus, suppose the displacement of A relative to B to be given by the line AB. We may resolve this displacement into components in any directions as those given by the arrows a and b by drawing lines from A and B parallel to these directions till they intersect at some point C. The lines AC and CB are the component displacements of AB in the required directions.

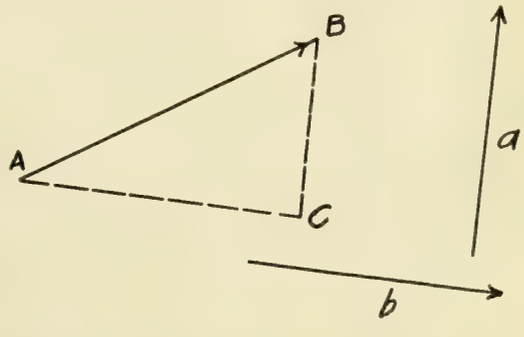


Fig. 7.

Velocity. — By velocity is meant the displacement of a point per unit of time. If a point moves from any initial position A to a final

position B in any path whatever and occupy t units of time in so doing: then the displacement AB divided by the units of time will be the mean velocity. Here appears the distinction between speed and velocity. Speed is the ratio of the distance passed over to the units of time taken up for the motion. Velocity is the ratio of the displacement to the time. Take the case of a point moving with uniform speed in a circle of radius r. Let t be the time of one complete revolution, then the mean speed is $2\pi r/t$. But the mean velocity in the time of one revolution is zero, since the displacement in that time is zero. In one half a revolution the mean speed is $\pi r \div t/2 = 2\pi r/t$, and the mean velocity for the same time has the magnitude $2r/t/2 = 4r/t$ and the direction is that of a diameter from the initial to the starting point.

The term, instantaneous velocity, is defined as the limiting magnitude and direction of the mean velocity when the interval of time is indefinitely small. In the limit, the distance passed over and the displacement coincide, hence at the limit the speed is the magnitude of the velocity, while its direction is tangent to the path

at the point at that instant.

Velocity may be uniform or variable. Uniform velocity has the same magnitude and direction irrespective of the interval of time. Hence it is the same as uniform speed in a straight line. When either the magnitude or direction changes, the velocity is variable. If a point is moving in a curve, it can not have uniform velocity. A point can be projected with the same speed in many different directions but its velocity can not be spoken of as the same velocity in different directions at the same time.

Velocity like speed has sign and magnitude. It also has direction and, hence, is a vector quantity.

Velocities may be resolved and combined the same as has been shown for displacements. Thus let a point move from P_1 to P_2 in any time t, the displacement is $P_1 a P_2$ and by defination the mean velocity $\equiv V = \frac{P_1 a P_2}{t}$. Let P_1 and P_2 be given by their coördinates x_1, y_1 and x_2, y_2; then the horizontal and vertical components of the mean velocity are

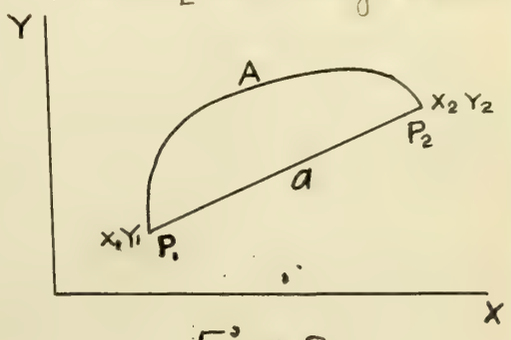


Fig. 8.

$$V_x = \frac{x_2 - x_1}{t}$$

and $V_Y = \frac{Y_2 - Y_1}{t}$

These equations give the horizontal and vertical components of the mean velocity. If P_2 approach indefinitely near P_1 and the interval of time becomes indefinitely small we shall have

$$V_X = \lim_{\Delta t = 0} \left[\frac{\Delta x}{\Delta t}\right] = \frac{dx}{dt}$$

$$V_Y = \lim_{\Delta t = 0} \frac{\Delta Y}{\Delta t} = \frac{dy}{dt}$$

In Fig. 8, if V_x is directed toward the right, we shall have x_1 numerically greater than x_2 in the 1st and 4th quadrants and positive. In the 2d. and 3d quadrants if V_x is directed toward the right x_2 is numerically greater than x_1 and negative in both cases. In all quadrants therefore V_x is positive when directed toward the right. In the 1st. and 2d. quadrants if V_Y is directed upward Y_2 is numerically greater Y_1 and both are positive. In the 3d. and 4th quadrants if V_Y is directed upwards Y_1 is numerically greater than Y_2 and both are negative. In all quadrants, then, V_Y is positive when directed upwards. Hence the rule, "If the direction of the line representative is toward the right, V_x is positive; if downward negative. If upward, V_Y is positive; if downward negative." This rule is general no matter whether we consider velocity in a

plane or in space.

Relative Velocity of Two Points. — Let M and N (Fig. 9.) be two points of a rigid system with the velocities represented respectively by vectors MR and NS. Suppose that the system in which M and N are moving to have a velocity equal but in the opposite sense to the velocity of N. Then N is subjected to equal but opposite velocities and, hence, will be brought to rest; while the point M will partake of the two velocities MR and MG (=NS). Therefore the resultant velocity of M is represented by the vector MH. We may consider this vector as representing the velocity which M has when N is at rest or the velocity of M relative to the point N.

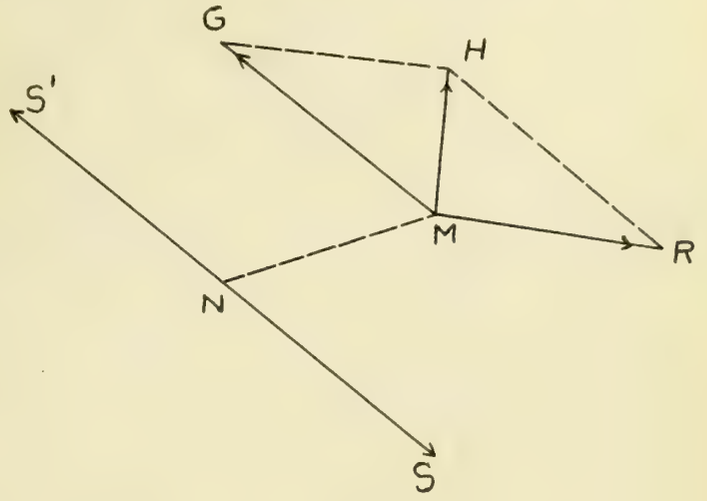


Fig. 9.

In the triangle MHR, the sides are vectors representing velocities as follows:

HR = NS represents velocity of point N.

MR = " represents velocity of point M.

MH " represents velocity of point M relative to point N.

By geometric addition we have

$$MR = MH + HR$$

or, the velocity of M is the geometric sum of the velocity of N and the velocity of M relative to N.

Also

$$MH = MR - HR$$

or, the velocity of M relative to N is the geometrical difference of the velocity of M and the velocity of N.

Take any point O, Fig. 10 and draw the vectors OM and ON to represent respectively the velocities of M and N, then

$$MN = ON - OM,$$

and $$NM = OM - ON.$$

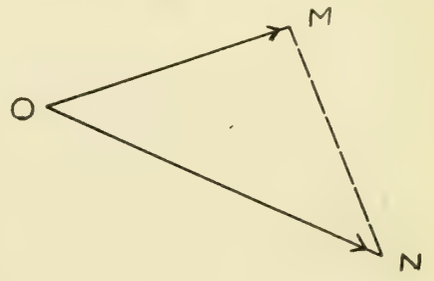


Fig. 10.

But MN represents the velocity of N relative to M and NM represents the velocity of M relative to N, therefore, the velocity of N relative to M is equal and opposite to the velocity of M relative N.

We have said that the velocity of M relative to N is the velocity of M, when the rigid system of which M and N are points, is so moved that N is brought to rest. Evidently in this case N is the instant center of the system. Since the velocity of any point is perpendicular to the instantaneous radius of the point, the velocity of M

relative to N must have a direction perpendicular to the line joining the points M and N. We may think of this in another way. If the relative velocity of M to N or N to M has a component along the line MN, the relative position of the two points must be changing; but M and N are points of a rigid system and can not change their relative positions. Hence the relative velocity can have no component along MN and must remain perpendicular to the line MN.

In Fig. 11, we have an example of the principles just stated. The connecting rod BC has a combined motion of translation and rotation. At the given instant P is the center about which the rod is rotating. Due to the rotation the point B has a velocity represented by the vector BF. Due to the translation, every point of the rod has a velocity represented by BF. The velocity of B relative to C is BF − BE = EF = BG. As has already been shown the relative velocity of B to C must be perpendicular to BC.

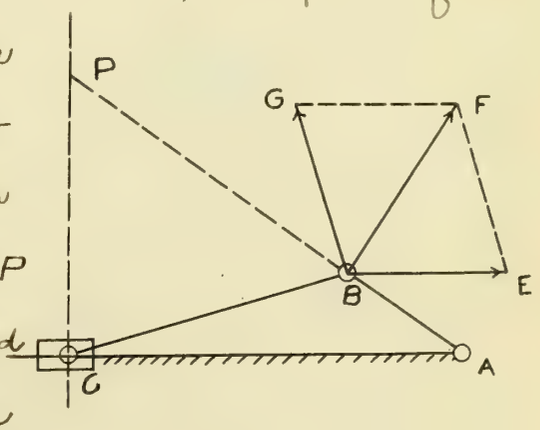


Fig. 11.

Velocity Images. — We shall next consider

three points of a rigid system. We will assume as known the velocity of two of the points. The problem is to find the velocity of the third point, which may be any point of the system.

Let A, B and C, Fig. 12. be the three points, and let the velocity of A and B be represented by the vectors AA' and BB' respectively. Note that in the case of a rigid system AA' and BB' are not independent.

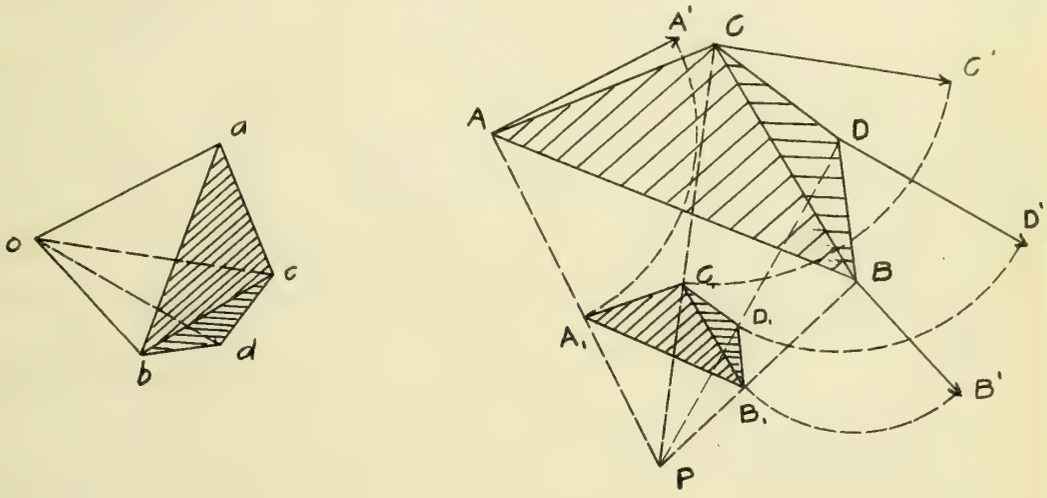


Fig 12.

Choose any point O as a pole and draw Oa and Ob equal to and parallel to AA' and BB' respectively. From what precedes ba represents the velocity of A relative to B, and is, therefore perpendicular to AB. Expressing this in another way we may say that the ends of the vectors representing the velocity of A and B lie on a line perpendicular to the line joining A and B. Similarly, will the ends of the vectors,

representing the velocities of A and C, lie upon a line perpendicular to AC. Also the ends of the vectors oD and oc will lie on a line through b and perpendicular to BC. Now since c is on these two lines it must be at their intersection. By drawing CC' equal and parallel to oc we have the velocity of C, which is any point of the system. The triangle ABC is similar to triangle abc, since by construction the sides of each are perpendicular to each other. We have then the following proposition: If from any point as a pole the vectors representing the velocities of the different points of a system are laid off, the ends of these vectors form a system similar to the original system.

To find the velocity of any point of the system we locate the corresponding point in the image and connect it with the pole o. Thus suppose we want to find the velocity of D (Fig. 12). We locate d in the image by remembering that d is in the same position relative to abc that D is relative to ABC. Then connect the point with the pole o and we have in od the vector representing the velocity of D.

The point o of the image corresponds to the point of the original system that has no velocity:

hence, making the triangle BPA similar to triangle boa we locate P, the instantaneous center of the system. It is to be remembered that the image abc is turned through 90° with respect to the original system. We may also find an image parallel to the original system. We know that the direction of the velocity of any point is perpendicular to the instantaneous radius of that point. By revolving the vectors, which represent the velocities, through 90° we make them coincide with the instantaneous radii. Thus in Fig. 12 the position of the vectors AA', CC' and BB' when revolved through 90° is AA_1, CC_1 and BB_1 respectively. If we connect the ends of these vectors by lines we shall have a figure similar to the original figure. Thus triangle $A_1B_1C_1$ is similar to triangle ABC since

$$AA_1 : BB_1 : CC = AP : BP : CP.$$

Hence the homologous sides are parallel. To find the velocity of any point, D, we draw DP and locate D_1 in the image then DD_1 is the required revolved velocity of the point D.

Example of the Application of Velocity Images.— In Fig. 13 is the skeleton of the McCord Quick Return Mechanism. AD and BC are two links, connected by a third link AB. From the joint B, the

the connecting rod runs off to block E. D and C are fixed centers, of which D is the center of the driving link AD. Choose any point o as a pole. Let

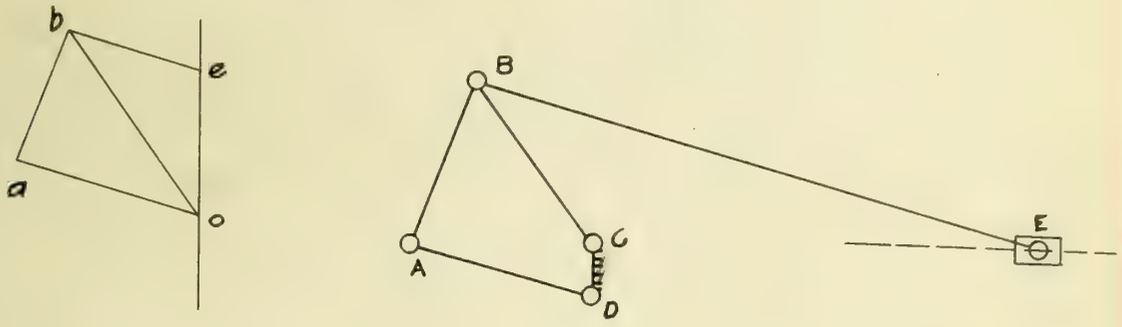


Fig. 13.

the velocity of the point A be represented for the instant by oa. Through o draw a line equal to and parallel to AD. We have already denoted this by oa. To find velocity of B we locate b in the image in the same relative position as it occupies in the mechanism. We do this by drawing through a and o lines parallel respectively to AB and BC. The distance ob represents the revolved velocity of B and ab is the image of the coupler AB. Again, draw a line through b parallel to BE. It will intersect a line through o perpendicular to line of travel of E in e. The distance oe represents the revolved velocity of E while be is the image of the connecting rod.

Acceleration. — Acceleration is the time rate

of change of velocity. Let P_1, P_2, P_3, etc. be the path of a moving point and let the instantaneous velocities at these points be v_1, v_2, v_3, etc. We know that the velocity is tangent to the path and is equal to the speed at that point. The mean speed of the point from P_1 to P_2 is path P_1P_2 divided by t. This ratio gives the units of speed. In Fig. 15, OP_1 is drawn to represent the velocity at P_1 and OP_2 to represent the velocity at P_2. The entire change of velocity between P_1 and P_2 is represented by the closing line P_1P_2 and the mean time rate of change of velocity is P_1P_2 divided by t. To this ratio the name acceleration is given. In this particular case, the acceleration is called the mean acceleration. Here the change in velocity is not in the direction of the velocity at either position of the point. If the point is moving in a straight line with an increasing speed and, therefore, an increasing velocity the change of velocity is in the direction of motion; but in general the change of velocity is not in the

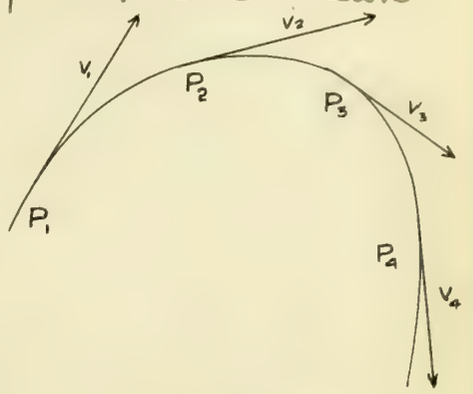


Fig. 14.

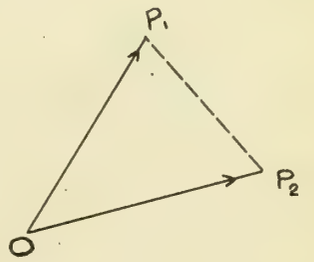


Fig. 15.

direction of motion.

In Fig. 15 let P_2 approach very near P_1, then we shall have for the limiting magnitude and direction of the mean acceleration, when the interval of time is indefinitely decreased, a new term to which the name instantaneous acceleration is given. In general, the term acceleration always means instantaneous acceleration. It may be defined as the limiting time rate of change of velocity, whether that change takes place in the direction of motion or not.

Acceleration may be zero, uniform or variable. If the acceleration is zero we shall have uniform speed and velocity in rectilinier motion. When the acceleration is uniform, it has the same magnitude and direction, whatever the interval of time. In this case the acceleration at any instant is equal to the mean acceleration. If the direction of acceleration coincides with the direction of velocity we shall have uniform rate of change of speed in a straight line. If these two do not coincide, the result is still uniform acceleration but the motion is curvilinear. In the case of variable accel-

eration either direction or magnitude or both change.

Acceleration in Curvilinear Motion :— In Fig. 16 suppose a point to be moving in the curve AB. When at A let its velocity be V_1 and when at B let its velocity be V_2. From any point O as a pole lay off ON to represent the velocity at A. Also from the same point lay off OM to represent the velocity at B. Join M and N. Then NM represents the change in velocity both in magnitude and direction between A and B, since vector OM is the resultant of ON + NM. If we divide this change of velocity by the time occupied in making the change we get an approximate value of for the rate of change of velocity, or the acceleration. Taking the limiting of this quotient we obtain an exact expression for the acceleration: thus

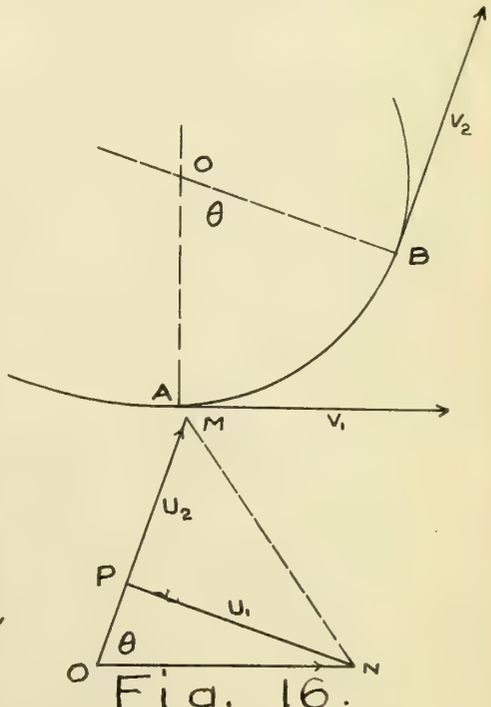


Fig. 16.

$$\text{Acceleration} \equiv f = \lim_{\Delta t \doteq 0} \left[\frac{MN}{\Delta t}\right]$$

We shall discuss further the acceleration as given by the preceding equation. We

may resolve the change of velocity into two components, one at right angles to OM and the other along OM. Let these components NP and PM [Fig. 16] be denoted by U_1 and U_2 respectively. Now as the point B approaches the point A the component U_1 will tend to have a direction at right angles to V_1 and the component U_2 will tend to have the direction of V_1. Finally, in the limit, i.e. when B reaches A the component U_1 of the change in velocity will be normal to the path AB, while the component U_2 will be tangent to the path AB. Hence the acceleration is resolved into two components, whose magnitudes are given by the limiting values of the ratios $\frac{NP}{\Delta t}$ and $\frac{PM}{\Delta t}$ respectively. We shall next express these limiting ratios in terms of known quantities

Angle ACB = angle NOB = θ

$$NP = ON \sin\theta = V_1 \sin\theta$$

$$\frac{NP}{\Delta t} = V_1 \frac{\sin\theta}{\Delta t}$$

$$\lim_{\Delta t = 0}\left[\frac{NP}{\Delta t}\right] = V_1 \lim_{\Delta t = 0}\left[\frac{\sin\theta}{\Delta t}\right]$$

$$= V_1 \lim_{\Delta t = 0}\left[\frac{\Delta\theta}{\Delta t}\right]$$

$$= V_1 \frac{d\theta}{dt}$$

But $\frac{d\theta}{dt} = \omega$, the angular velocity of the point

about C. Also $v_1 = \rho\omega$ or $\omega = \frac{v_1}{\rho}$. Hence

$$\lim. \frac{NP}{\Delta t} \equiv f_n = v_1 \frac{d\theta}{dt} = v_1 \omega = \frac{v_1^2}{\rho}$$

From Fig. 16 we have

$$\lim. \frac{PM}{\Delta t} = \lim. \frac{v_2 - v_1}{\Delta t} = \frac{dv}{dt} = \frac{d^2s}{dt^2} \equiv f_t.$$

Since the components are at right angles, the total acceleration $\equiv f$ is

$$f = \sqrt{f_n + f_t} = \sqrt{\frac{v_1^2}{\rho} + \frac{d^2s}{dt^2}}$$

In case the point is moving uniformly along the curve we shall have the derivative of the velocity with respect to the time equal to zero. Therefore, the total acceleration is given by the radial component $\frac{v^2}{\rho}$.

Part II.

Acceleration in a Plane.

Rigid System — Thus far we have discussed the motion of a point along any curve. Next we shall take up the study of the motion of a rigid system. Such a system is one in which the points that form the system do not change their positions relative to each other. In reality such a system does not exist; but for the purpose of a kinematic analysis, we may consider the points forming the system as being immovable relative to each other.

If a rigid system has a motion parallel to a fixed plane, the motion is completely represented by the motion on any plane section of the body in its plane; hence in dealing with cases of plane motion, we shall consider the system as represented by a plane figure lying in the plane of the paper. The position of the plane figure is determined by the position of any line lying in the plane of the figure; hence the motion of the figure in its plane is com-

pletely determined by the motion of two of its points, or by the motion of a straight line lying in the plane figure.

Rotation about a fixed axis. — If a plane system of points rigidly connected, rotates about a fixed ~~point~~, that point is the center of the paths of all points of the system. The normal and tangential components of the acceleration of any point A, [Fig. 17] at a distance r from the center are, respectively

$$f_n = \frac{v^2}{r} = r\omega^2$$

$$f_t = \frac{dv}{dt} = r\frac{d\omega}{dt}$$

where ω denotes the angular velocity of the system. In Fig. 17, O is the fixed point and A any point of the rotating system. The components of A's acceleration are

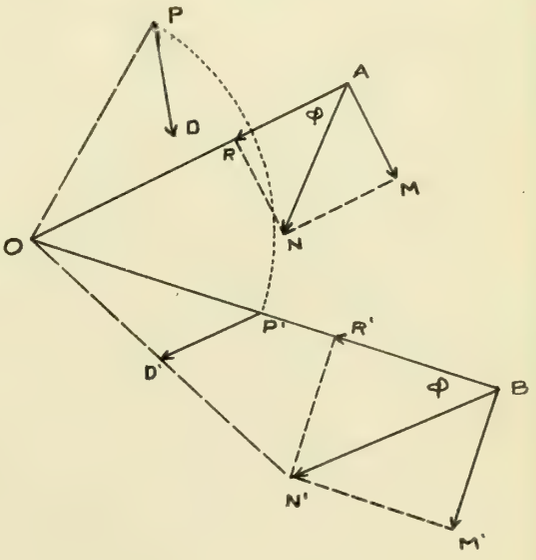


Fig. 17.

$$AM = OA\,\frac{d\omega}{dt}.$$

and

$$AR = OA\,\omega^2.$$

The resultant acceleration of A is AN, or

$$AN = \sqrt{\overline{AM}^2 + \overline{AR}^2}$$

$$= OA\sqrt{\left(\frac{d\omega}{dt}\right)^2 + \omega^4}$$

For any other point B we shall have for its

resultant acceleration

$$BN' = \sqrt{\overline{BM'}^2 + \overline{BR'}^2}$$

$$= OB\sqrt{\left(\frac{dw}{dt}\right)^2 + w^4}$$

The acceleration AN makes the angle $\phi = \tan^{-1}\frac{NR}{AR} = \frac{dw}{dt} \div w^2$. Also the acceleration BN' makes the angle $\phi_1 = \tan^{-1}\frac{N'R'}{A'R'} = \frac{dw}{dt} \div w^2$. Evidently $\phi_1 = \phi$; hence the resultant acceleration of a point rotating about a fixed point makes a constant angle with the radius vector and is proportional to the distance of the point from the fixed point.

If we know the acceleration of any one point of the system, the acceleration of any other point is readily obtained. Thus in Fig. 17 let BN' represent the known acceleration of the point B, we are to find the acceleration of any point, say P. Rotate P to P' on OB. Draw the line ON', also draw line P'D' parallel to A'N' [D' lies in ON']. Then is P'D' the line representing the acceleration of P. Constructing triangle OPD equal to OP'D', we have in PD the required acceleration.

Translation.— Let a rigid plane system have a motion of translation either curvilinear or rectilinear; then all points of the system have precisely the same motion, therefore the same velocity and acceleration. When the transla-

tion is rectilinear, the direction of acceleration is that of the motion. In the general case, however, in which all points move in parallel curves, the direction of the acceleration will make some angle with the tangent to the path. For example, the acceleration of any point on the parallel rod of a locomotive will be parallel to the acceleration of the crank pin. With constant speed of pin, the acceleration will be directed toward the center of driver.

Combined Motions.— Sometimes it is desirable to know the acceleration of a machine part which has compound motion. By compound motion as here used is meant the combinations of any two simple motions. It is convenient to consider the absolute motion of such machine-part or system as composed of two motions: (1) the motion of the first system relative to a second moving system: (2) the motion of the second system relative to a fixed system. As an example, take the motion of the block of the Whitworth Quick Return mechanism. We may consider the motion of the block as though the crank were at rest and then consider the motion of the crank relative to the fixed link or frame. We may combine the

two motions and get the resultant absolute motion relative to the frame. Hence any point in the given system S_1, may therefore be considered as moving in a curve of a second system S_2, while the curve at the same time moves in the fixed system Σ. It can be easily shown that the absolute velocity of a point is the resultant of the velocity in the curve and the velocity of the point of the curve which at the instant coincides with the given moving point. In Fig. 18, the point P moves in the curve m with a velocity U while the point of the curve coincident with P has a velocity W relative to the fixed system. If the curve m were at rest, the point P would have the velocity U alone, while if the point P were fixed on the curve it would have a velocity W alone due to the motion of the curve.

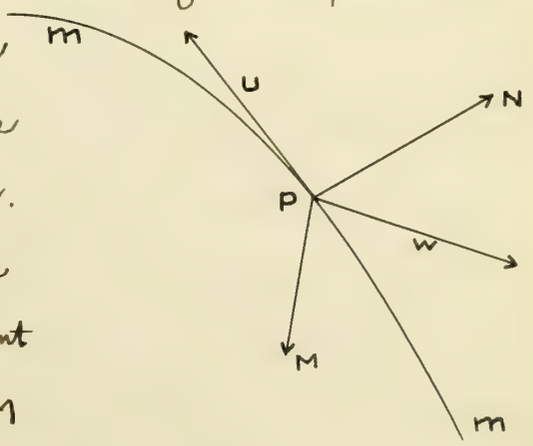


Fig. 18.

Consider now the actual acceleration of the moving point P, Fig. 18. Let the vector PM represent the acceleration of P when the curve m is at rest, and the vector PN represent the acceleration of the point of m coincident with P when the curve has the velocity W.

If the same law holds for accelerations as velocities, the actual acceleration of P is the resultant of the relative acceleration and the acceleration of the coincident point. We are now to determine if such is the case. We shall consider two cases: (1) when the curve m has a motion of translation either rectilinear or curvilinear; (2) when the curve has any motion in a plane.

Motion of the curve is a Translation. — In Fig. 19, m_1 and m_2 are the initial and final positions of the curve m corresponding to some time interval Δt. The point P is at P_1 on m_1 and has a velocity represented by U_1, while the coincident point of the curve has a velocity represented by W_1. In the time Δt the curve m will have moved to m_2 and the corresponding velocities will be U_2 and W_2. From O, lay off $OA = U_1$, and from A lay off $AB = W_1$: the closing line OB represents the actual velocity of the point P when at P_1. Similarly lay off $OC = U_2$ and $CD = W_2$. As before OD represents the actual velocity of the point P when at P_2. The change of velocity experienced by the point P in passing from P_1 to P_2 is represented by BD. Therefore, the acceleration of P is

$$f = \lim_{\Delta t = 0} \left[\frac{\Delta V}{\Delta t}\right] = \lim_{\Delta t = 0} \left[\frac{BD}{\Delta t}\right]$$

BD is the resultant of two component velocities, each of which gives rise to a definite accelerations

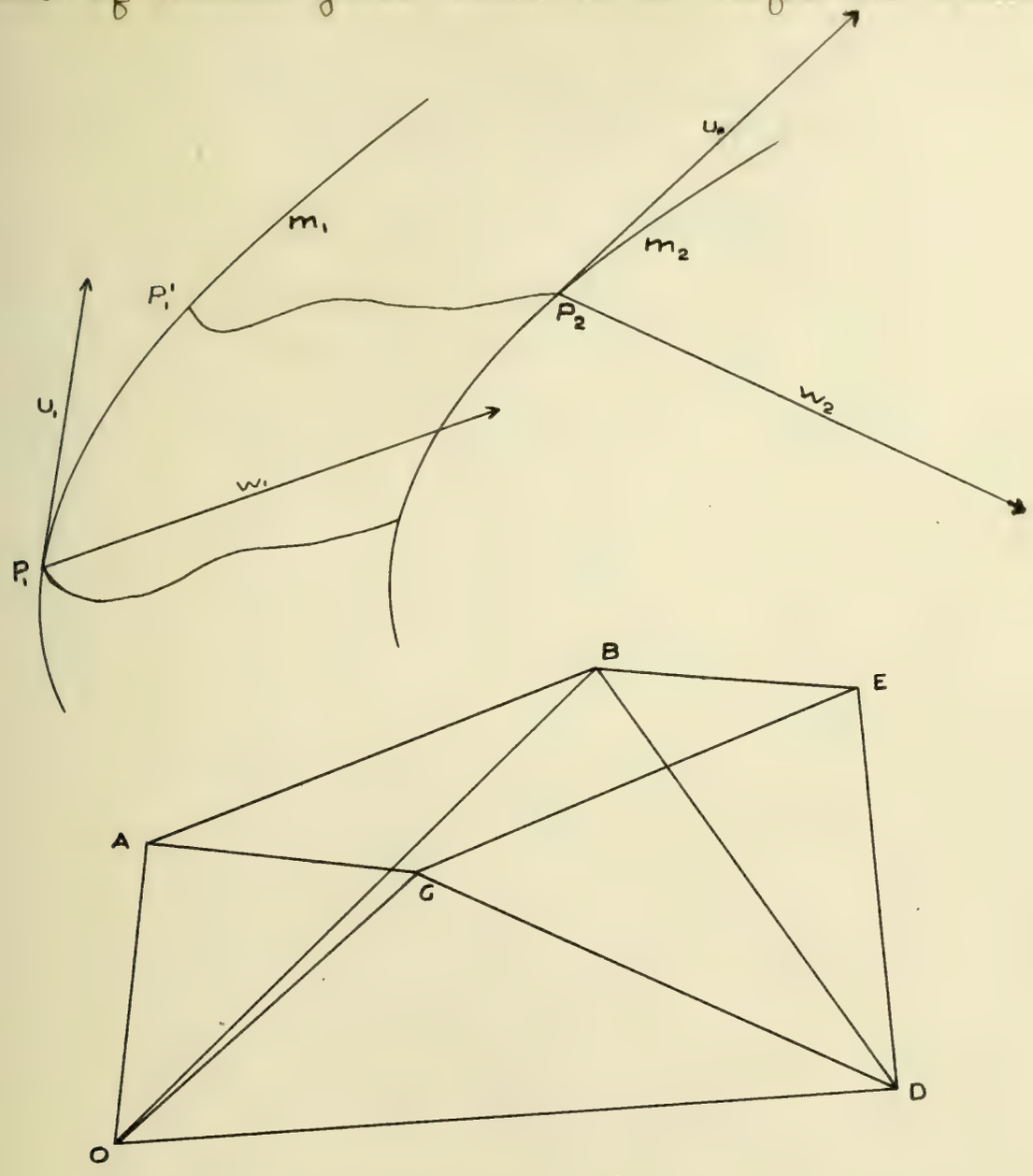


Fig. 19.

We shall next find these accelerations of the component velocities. From C draw CE parallel to AB and from B draw BE parallel to AC: join E and D. Geometrically, then BD = BE + ED = AC + ED and

$$\lim_{\Delta t \doteq 0}\left[\frac{BD}{\Delta t}\right] = \lim_{\Delta t \doteq 0}\left[\frac{AC}{\Delta t}\right] + \lim_{\Delta t \doteq 0}\left[\frac{ED}{\Delta t}\right].$$

The vector AC is the closing side for the vectors representing the velocities U_1 and U_2, which the point P would have, were the curve m at rest. Hence $AC = U$ and

$$\lim_{\Delta t = 0} \left[\frac{AC}{\Delta t}\right] = \lim_{\Delta t = 0} \left[\frac{\Delta U}{\Delta t}\right]$$

The second member of the preceding equation is the acceleration of the point P due to its motion in the curve m. The vector ED is the closing side for the vectors which represent the velocities of the points of the curve m coincident with P. By construction, CD represents the velocity w_2 of the point of the curve m coincident with P_2. In the second position of the curve the point originally at P_1 is at P_1' and as the curve has a motion of translation the velocity of P_1' of the curve is the same as that of the point P_2 of the curve. This velocity is represented by the vector CD (= AB). From this consideration, it follows that

$$\lim_{\Delta t = 0} \left[\frac{ED}{\Delta t}\right] = \lim_{\Delta t = 0} \left[\frac{\Delta w}{\Delta t}\right]$$

The second member of the preceding equation is the acceleration of the point of the curve m coincident with P.

We shall denote by f_r, the relative acceleration, that is, the acceleration of the motion of P relative to the path m: and by f_m, the

acceleration of the point of the curve coincident with P. We have, therefore,

$$f_r = \lim_{\Delta t \doteq 0} \left[\frac{AG}{\Delta t}\right] = \lim_{\Delta t \doteq 0} \left[\frac{BE}{\Delta t}\right]$$

also $$f_m = \lim_{\Delta t \doteq 0} \left[\frac{ED}{\Delta t}\right]$$

and $$\lim_{\Delta t \doteq 0} \left[\frac{BD}{\Delta t}\right] = \lim_{\Delta t \doteq 0} \left[\frac{BE}{\Delta t}\right] + \lim_{\Delta t \doteq 0} \left[\frac{ED}{\Delta t}\right]$$

or $$f = f_r + f_m .$$

We have, therefore, the following principle when the curve m has a motion of translation only: The acceleration of the point P is the geometrical sum of the relative acceleration of P in the curve m and the acceleration of the coincident point of the curve.

There are many applications of the above law in existing machines. The parallel rods of a locomotive, the compound pulley, certain gearing on hoisting machinery are all examples. It will be observed that each of these is similar to a disc rolling on a straight line in a plane. In such a case, the point P of the disc actually describes the arc of a cycloid [see Fig 20] but we may consider that P moves in a circle m which at the same time has a motion of translation

parallel to the plane. [AB is the trace]. Let the center O move with constant velocity w; then P moves with constant speed in the circle m while the circle has a constant velocity w of translation, parallel to AB. Under these conditions, the relative acceleration is $\frac{U^2}{OP}$ and is directed toward the center O. The acceleration of the point of the curve is zero. Hence the actual acceleration of P is

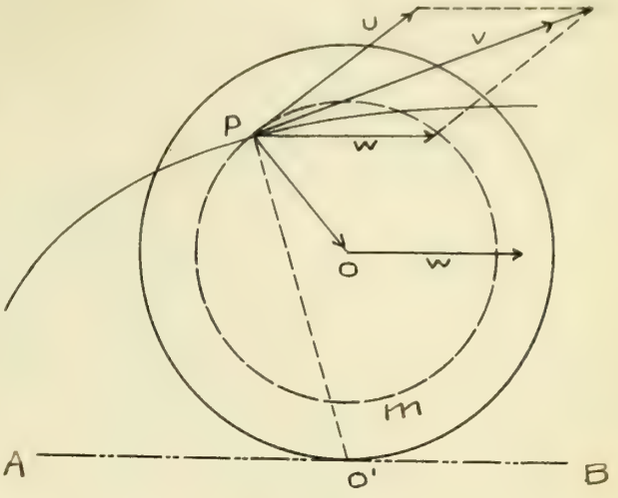


Fig. 20.

$$f = f_r + f_m = \frac{U^2}{OP} + 0 = \frac{U^2}{OP}.$$

which is the same as if the disc were rotating with the same angular velocity about O as a fixed center.

Motion of the curve is general. Coriolis Law. — In Fig. 21, m_1 and m_2 represent the consecutive positions of the curve m. O is the instant center of the motion of m. [System S_1] A point at P_1 of the curve has a velocity w_1 and the point at P_1 [system S_2] moves in the curve m with a velocity u_1. In the second position of the curve m the point P is at P_2 and has the velocity w_2, due to the motion of the curve and

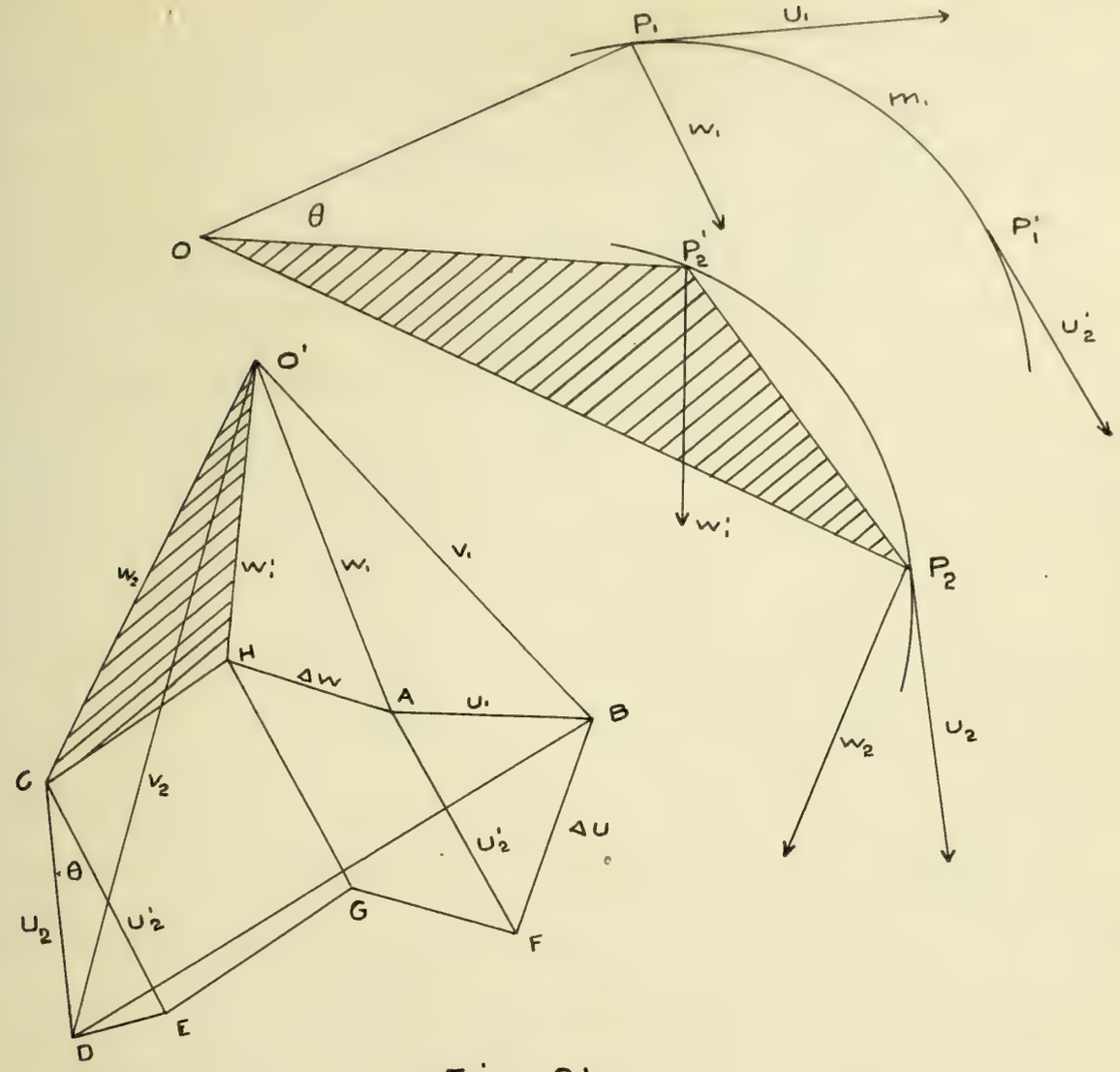


Fig 21.

a velocity U_2 due to the motion in the curve. In the figure O'A is laid off to represent W_1 and AB to represent U_1. The closing side OB then represents V_1, the velocity of the moving point relative to the fixed system Σ. Likewise OC and OD, equal to W_2 and U_2 respectively, are laid off. The closing side OD represents the velocity V_2 of the point when at P_2. Join B and D and we have the entire change in

velocity during the motion. We have therefore

$$f = \lim_{\Delta t = 0} \left[\frac{BD}{\Delta t}\right]$$

Let us first consider the acceleration of the point of the curve at P_1. The initial velocity of this point is w_1 and the final velocity of the point of the curve coincident with the moving point P is w_2. When the curve m is at m_2, P_1 is at P_2'. In order to find the velocity w_1 we have to construct the triangle O'GH similar to triangle $OP_2'P_2$. Then from the theory of velocity images the side O'H represents the velocity of the point P_1 of the curve when in the position P_1. As a result the vectors O'A and O'H represent the initial and final velocities of the point P of the curve m. The side AH, therefore, represents the change in velocity which the point P of the curve has experienced in passing from P_1 to P_2'. The acceleration due to this change of velocity is

$$f_m = \lim_{\Delta t = 0} \left[\frac{AH}{\Delta t}\right]$$

We shall consider now the acceleration which the point would have, if the curve m had remained at rest. The consecutive position of the point P would have been P_1' instead of P_2. Its velocity therefore would be v_2' instead of v_2. It is to be remembered that the velocity in the curve m is the same no matter whether the point is at P_1' or P_2. Through C draw

CD equal to v_2. Through same point C draw CE parallel to v_2' and equal to CD. From A lay off AF equal to and parallel to CE. In the triangle ABF we have the sides AB and AF representing the initial and final velocities v_1 and v_2' of the point in the curve m. The closing line BF, therefore, represents the change in velocity which the point would undergo had the curve remained at rest. Hence the acceleration is

$$f_r = \lim_{\Delta t = 0} \left[\frac{BF}{\Delta t} \right]$$

By construction AF and CE are parallel and equal. From F let a line be drawn parallel to AH and from E one parallel to CH. These two lines intersect in G. It is easily seen that AH = FG, and HC = GE. Geometrically then

$$BD = BF + FG + GE + ED.$$

The quantity BD as we have seen represents the total change in velocity of the point P. It is the sum of four components, two of which we have already determined. We have, therefore, to determine the remaining two, GE and ED.

In the triangle O'CH, HC represents the velocity of P_1 relative to P_2'. Hence, if ω_2 denotes the angular velocity of the curve when in position m_2 we have

$$HC = \omega_2 \times \text{chord } P_2' P_2$$

and
$$\lim_{\Delta t \to 0}\left[\frac{HC}{\Delta t}\right] = \omega_2 \lim_{\Delta t \to 0}\left[\frac{\text{chord } P_2' P_2}{\Delta t}\right]$$
$$= \omega_2 \lim_{\Delta t \to 0}\left[\frac{\text{arc } P_2' P_2}{\Delta t}\right]$$
$$= \omega_2 u_2$$

In the triangle CDE the angle $DCE \equiv \theta$ is equal to angle $P_2' O P_1' =$ angle $P_1 O P_2'$ through which the curve m has turned in the time Δt. Therefore,

$$DE = 2 \times CD \times \sin \tfrac{1}{2}\theta.$$
$$= 2 u_2 \sin \tfrac{1}{2}\theta.$$

and
$$\frac{DE}{\Delta t} = 2 u_2 \frac{\sin \frac{1}{2}\theta}{\Delta t}$$

or
$$\lim_{\Delta t \to 0}\left[\frac{DE}{\Delta t}\right] = u_2 \lim_{\Delta t \to 0} \frac{2 \sin \frac{1}{2}\theta}{\Delta t}$$
$$= u_2 \frac{d\theta}{dt}$$
$$= u_2 \omega_2$$

It thus appears that $\frac{HC}{\Delta t}$ and $\frac{DE}{\Delta t}$ have the same limiting value $u_2 \omega_2$. The next thing is to find out the limiting directions of each. In the limit P_2 approaches P_1 and the chord $P_2' P_2$ becomes the tangent at P_2'. But HC must always remain perpendicular to chord $P_2' P_2$. Hence in the limit HC is normal to the curve at P_1', that is, perpendicular to the relative velocity u_1. Also w_1' becomes w_1 in the limit and the angle θ becomes zero while DE becomes perpendicular to the relative velocity u_1. Since these components have the same limiting direction we may add

algebraically. We have then

$$\lim_{\Delta t \doteq 0}\left[\frac{BD}{\Delta t}\right] = \lim_{\Delta t \doteq 0}\left[\frac{BF}{\Delta t}\right] + \lim_{\Delta t \doteq 0}\left[\frac{FG}{\Delta t}\right] + \lim_{\Delta t \doteq 0}\left[\frac{GE}{\Delta t}\right] + \lim_{\Delta t \doteq 0}\left[\frac{ED}{\Delta t}\right].$$

$$= f_r + f_m + v_2\omega_2 + v_2\omega_2,$$

or, in general,

$$f = f_r + f_m + 2v\omega,$$

dropping the subscripts upon passing to the limiting condition. The component $2v\omega$ is the additional acceleration which the point experiences, due to the fact that the system S_1 has any motion in a plane rather than a translation. This component is sometimes called the Coriolis Component or Compound Supplementary Acceleration.

We have seen that the direction of this component is perpendicular to that of the relative velocity: the sense, however remains to be determined. In Fig. 21, as GE and DE approach their limiting conditions, that is as DB becomes infinitely small, the sense of each is downwards. D will lie below E, and E below G. Therefore, the sense of $2v\omega$ is downwards. In any case the sense of the supplementary acceleration is such that if applied to the arrowhead of the velocity v_1, it would tend to

rotate the arrow, representing the relative velocity U_1, about the instant center O in the sense of the angular velocity ω.

We have now the following principle when the curve m has any motion in a plane: The acceleration of the point P is the geometrical sum of the relative acceleration of P in the curve m, the acceleration of the coincident point of the curve, and two times the relative velocity of the point P into the angular velocity of the system Σ_1,

Applications of Coriolis' Law — In Fig. 22, a line rotates about one of its points O, and a point P moves with a velocity U along the line. Let us find the acceleration of the point P along and perpendicular to the line OP. The point P of the line has the components $r\omega^2$ directed toward O, and $r\frac{d\omega}{dt}$ perpendicular to OP. Since the direction of the motion of P is always along the line, we have for the relative acceleration the quantity $\frac{dU}{dt}$ along the line. The component $2U\omega$ is perpendicular to OP. and has, therefore, the same direction as the tangential component $r\frac{d\omega}{dt}$. Hence the

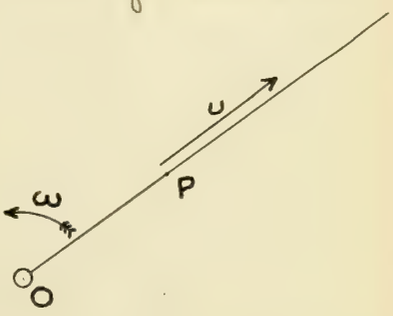


Fig. 22.

resultant acceleration along OP is

$$f_1 = \frac{dv}{dt} + (-r\omega^2),$$
$$= \frac{dv}{dt} - r\omega^2.$$

and that at right angles to OP is

$$f_2 = 2v\omega + r\frac{d\omega}{dt}.$$

But $v = \frac{dr}{dt}$: and $\frac{dv}{dt} = \frac{d^2r}{dt^2}$

Also $\omega = \frac{d\theta}{dt}$: and $\frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$

Substituting these values in equations above we have

$$f_1 = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$$

$$f_2 = 2v\omega + r\frac{d^2\theta}{dt^2}$$

Fly Wheel Governor.— A heavy mass is pinned to the arm of a fly-wheel at the point O' [Fig. 23] A point P of the mass, therefore, moves in a circular path with O' as a center. The wheel itself rotates about O with an angular velocity ω. Let ω_1 denote the angular velocity of the point P relative to O': then the velocity of the point P in its curve is

$$v = \overline{O'P}\,\omega_1$$

and the acceleration due to rotation about O' has the two components

(1) $\frac{v^2}{O'P} = \frac{O'P \cdot O'P\,\omega_1^2}{O'P} = O'P\,\omega_1^2$

(2) $O'P\frac{d\omega_1}{dt}$

directed toward O' along O'P, and at right angles to O'P

The point P of the wheel has the two components

(3) $OP\,\omega^2$ directed toward O.

(4) $OP\,\frac{d\omega}{dt}$ at right angles to OP.

In addition to these four components, there

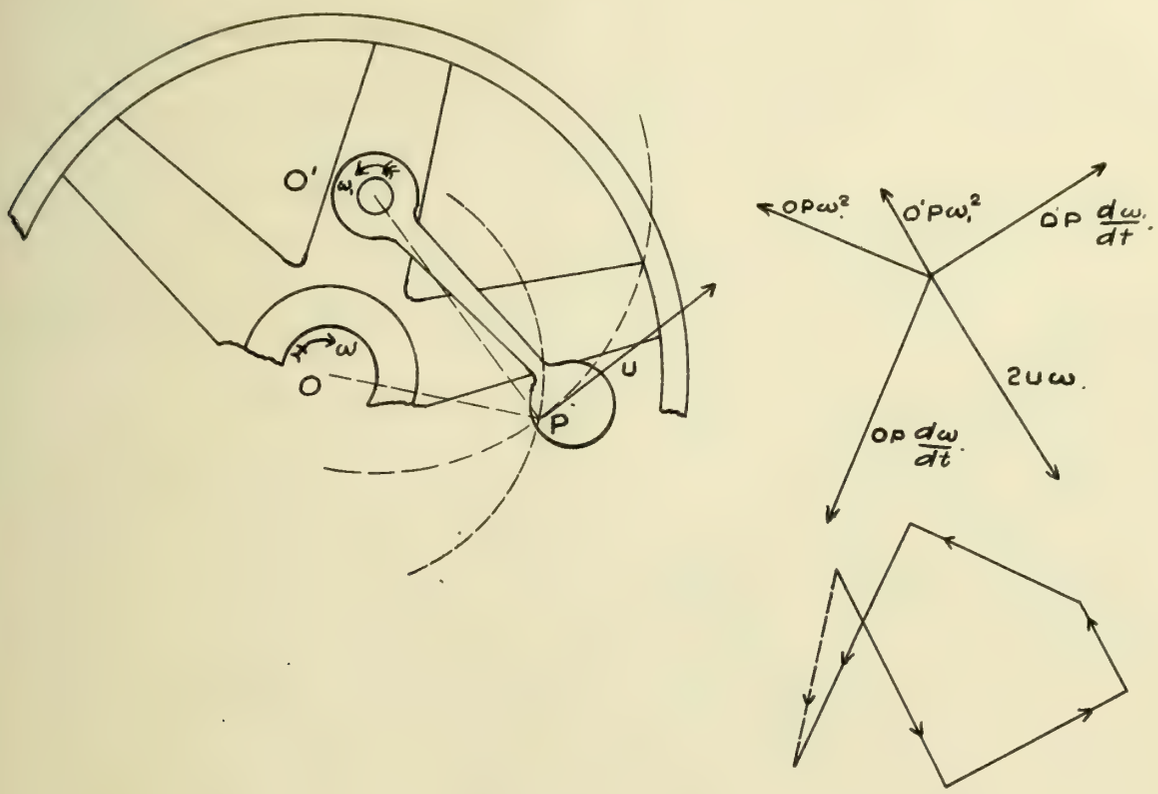


Fig. 23.

is the component $2U\omega$. Its direction is along O'P and its sense opposite to the component $O'P\omega_1^2$. Laying off the five components in proper, we have in the closing side of the polygon the resultant acceleration of the point P.

Relative Acceleration of Two Points.— When a plane system has any motion in a plane, this motion may be resolved into a translation and a rotation. We have already shown that the actual acceleration of any point is the resultant of two components, one due to the translation and one due to the rotation. The translation will give all points of the system the same velocity, consequently the same acceleration. The rotation will give all points of the system, accelerations proportional to their distances from the instant center. In Fig. 24 let AK represent the the acceleration of A due to the translation, then all points as C and B will have accelerations equal to and parallel to AK. Let the vector BN represent the acceleration of the point B due to the rotation then since A is the instant center of the system; the vector CH will represent the acceleration of C due to the rotation. To find the acceleration of any point as D we merely

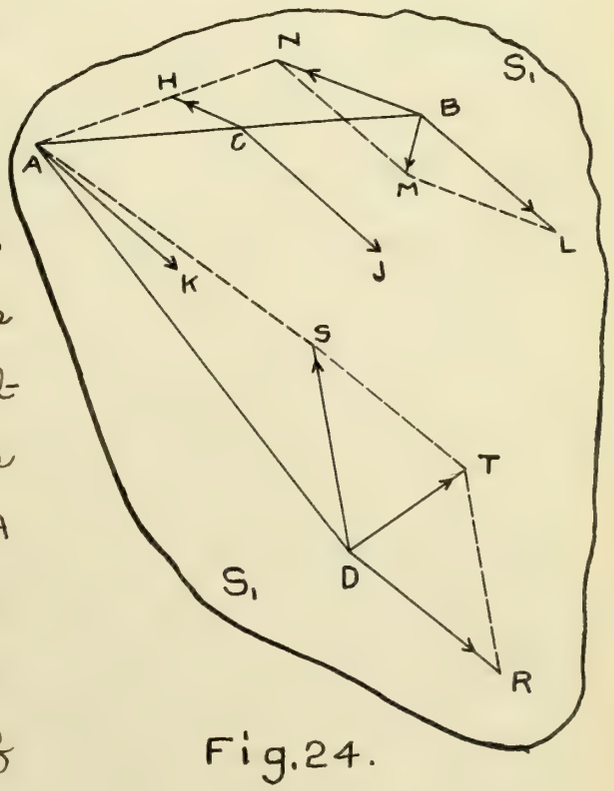


Fig. 24.

construct triangle ADS similar to triangle ABN. We can readily obtain the total acceleration of D by laying off DR equal to AK and completing the parallelogram upon DS and DR. The vector DT is the resultant acceleration of D.

From the preceding paragraph, it is evident that if the acceleration of any two points of a system are known, we can readily determine the acceleration of any other point. Let us consider further the acceleration of A and B in Fig. 24. We have assumed that the motion of the system S, is resolved into a motion of translation and rotation about A as a center. In this case, therefore, the acceleration of any point is the resultant of two components. Due to the translation, A has an acceleration represented by the vector AK. Since A is the instantaneous center, the acceleration component due to the rotation must be zero. Hence it follows, that the translation component is the total acceleration of A. We may assume that the total acceleration of B is represented by the vector BM. But B has also motion of translation the same as A, hence its acceleration due to this motion is equal and parallel to AK. By geometric subtraction we have

$$BM - BL = -ML = +BN.$$

The vector BN represents the acceleration of B due to the motion of rotation about A. Knowing this component, we can determine the acceleration of any other point as has already been shown.

We may describe the component BN as the acceleration relative to A since this component is due to the motion of rotation; and also since A's acceleration due to the rotation is zero. In the triangle BML we have, by geometric addition,

$$BM = BL + LM$$
$$= BL + BN$$

which we may interpret as follows: The acceleration of B is the geometric sum of the acceleration of A and the acceleration of B relative to A.

In Fig. 25, let the vectors AK, BN, BM, and BL be drawn as in Fig. 24. Project these vectors upon the line AB. Now since AK, MN, and BL are equal and parallel lines, their projections on the line AB are equal, that is $AK' = N'M' = BL'$. Considering segments in the direction AB as positive, we have AK' as

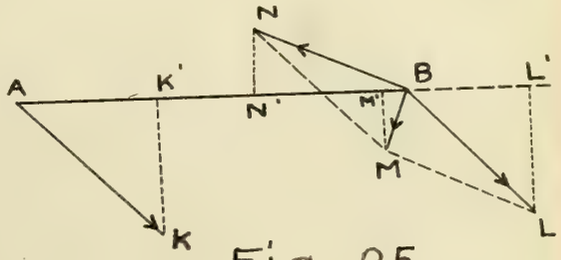


Fig. 25.

positive and BM' and M'N' as negative segments. Hence

$$AK' = N'M' = N'B - M'B = -BN' - [-BM']$$
$$= BM' - BN'$$

or, the projection of A's acceleration upon the line AB is equal to the projection of B's acceleration minus the projection of the acceleration of B relative to A.

The vector BM which represents the total acceleration of B may be resolved into components, one perpendicular to AB and another along AB, toward center A. Evidently BM' is the radial component of BM. The magnitude of any radial component is $\frac{v^2}{r}$ where v is the linear velocity and r the radius; hence

$$BM' = \frac{[\text{Velocity of B relative to A}]^2}{\text{length AB.}}$$

Klein's Construction.— Let NV, Fig. 26. represent the velocity of N relative to M. With NV as a radius describe a circle. Upon MN as a diameter describe a second circle. Draw the common chord RP. It will intersect MN in D and ND is the radial component

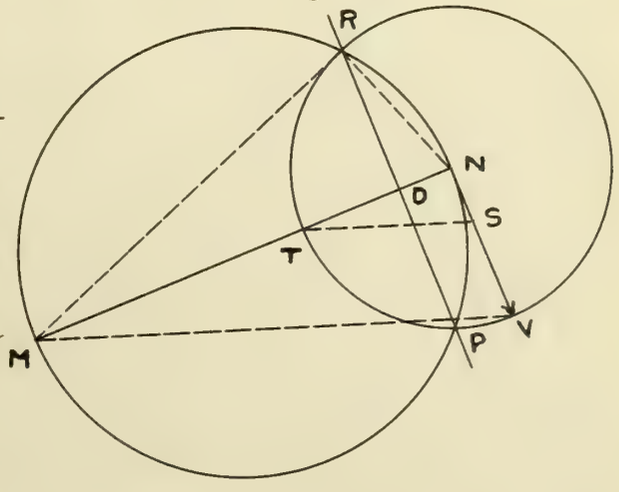


Fig. 26.

of N's acceleration relative to M. The proof is as follows: From the similarity of the triangles MRN and RDN we have

$$\frac{ND}{RN} = \frac{RN}{MN}.$$

or

$$ND = \frac{\overline{RN}^2}{MN} = \frac{\overline{NV}^2}{MN}.$$

$$= \frac{v^2}{r}.$$

A second construction may be obtained. Lay off NT equal to NV. Draw MV and through T draw a line parallel to MV. Then will NS = ND be the required radial component of B's acceleration relative to A. For, from the similarity of the triangles TNS and MNV we have

$$\frac{NS}{NV} = \frac{NT}{NM}.$$

or

$$NS = \frac{\overline{NV}^2}{MN} = \frac{v^2}{r}$$

Four Link Mechanism:— In a four link mechanism, the acceleration of one end of the coupler is known; and the acceleration of the other end is required. In Fig. 27, DE represents the acceleration of the point D of the crank AD. Since the point D is common to both the crank AD and the coupler DC the acceleration of one point of the coupler is known, if the acceleration of the point D of the crank AD is given. Project E upon AD in E'. The length DE' is the radial component of D's acceleration. Its magnitude, therefore, is $\frac{v^2}{r}$

To find the velocity V, we construct a semi-circle upon DA; then DF = DF' is the required velocity. For, from the similarity of the triangles DFE' and DFA we have

$$DE' : DF = DF : AD$$

or

$$DE' = \frac{\overline{DF}^2}{AD} = \frac{V^2}{r}$$

To find the velocity of C, we first locate the instant center, P, of the link DC. Then since the velocity

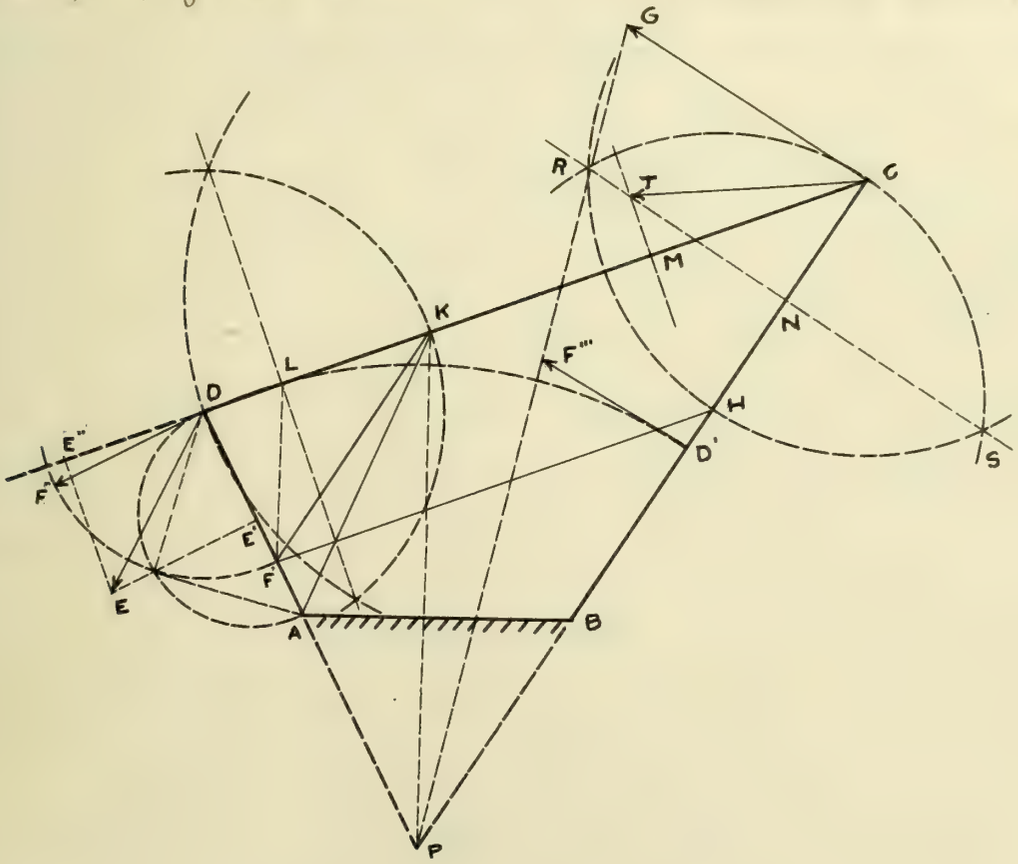


Fig. 27.

of all points of a rotating system are proportional to their distances from the instant center, we have, by transferring DF'' to D'F''' and constructing triangle PGG similar to triangle PD'F''', in the vector

CG the velocity of the point C. Another method of finding the velocity of the point C is as follows: Draw F'K parallel to BC and F'H parallel to DC. Then F'K = HC is the revolved velocity of C. [See paragraph on velocity images for proof.] In the triangle DKF', the sides DF' and KF' represent the revolved velocities of the points D and C respectively. Hence DK represents the velocity of D relative to C. By Klein's construction, the radial component DL is found. We may also find this component by connecting the point K with the instant center P, and drawing from F', the line F'L parallel to PK. Then will F'L intersect DK in L. The proof is as follows: Triangles DLF' and DKP are similar, as also are triangles DKF' and DCP. We may, therefore, write the following

$$\frac{DL}{DF'} = \frac{DK}{DP},$$

from which $$DL = DK \times \frac{DF'}{DP};$$

but $$\frac{DF'}{DP} = \frac{DK}{DC},$$

Therefore $$DL = \frac{\overline{DK}^2}{DC};$$ which is the same as we get by using Kleins construction.

Project DE upon DC in E". From preceding considerations, we know that the projection of C's acceleration is equal to the projection of D's acceleration, minus (geometrically) the projection of the acceleration

of D relative to C. The projection of C's acceleration is, therefore, LE". Laying off this distance LE" from C, we have the segment CM = LE". The end of the vector representing the acceleration of C must lie on the perpendicular to CD drawn through M. The velocity of C relative to D is given by the segment CH. Employing Klein's construction, we readily find the radial component CN of C's acceleration toward B. The extremity of the vector representing the acceleration of C must lie somewhere on the chord RS, but it also lies upon the perpendicular to CD at M. Hence it lies at their intersection in T and CT represents the acceleration of C to the same scale that DE represents that of D.

The foregoing discussion is general. We shall next consider a special case of the four link chain in which the crank AD is revolving uniformly. The equations for the radial and normal accelerations of a point are

$$(1)\quad f_n = r\omega^2$$

$$(2)\quad f_t = r\frac{d\omega}{dt}$$

In the present case, the angular velocity is constant, therefore $\frac{d\omega}{dt} = 0$. Consequently, the normal component, f_t, is zero. We may then take the crank length AD to represent both the accelera-

tion and resolved velocity of D. Employing the same methods as in the preceding case, we readily find the resultant velocity of C to be CH and the radial acceleration of D toward C to be DL. The projection of D's acceleration on DC is DA' and the projection on the same line of the acceleration of D relative to C is DL, hence the projection of C's acceleration upon DC is DA' − DL = LA'. Laying this distance LA' off from C we have for C's projection the segment CM. As in the previous case the vector representing C's acceleration must lie somewhere in the perpendicular to BC through M. Employing Klein's construction, we find the projection of C's acceleration on CB to be CR. Therefore, the end of the vector representing C's acceleration lies in the perpendicular to BC at R or in RT. But the end of the vector also lies in the perpendicular MT, hence it lies in the intersection of MT and RT at T. The vector CT, therefore, represents the acceleration of C to the same scale that DA represents the acceleration of D.

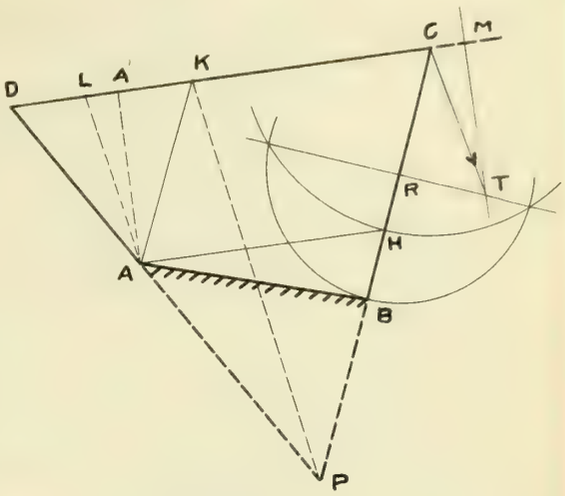


Fig. 28.

It may happen, that when the crank length

DA is taken to represent the acceleration of D that the resolved velocity of C falls beyond B. In fig. 29, we have such a condition illustrated. In this case, Klein's construction fails, since the circles do not intersect. We can, however, find the radial component of C's acceleration toward B. We shall assume that the points H, K, L, M, and A have been located as in the preceding cases. Draw any line CX through C and join any point of this line as Y to B and H. Draw HW parallel to BY and RW parallel to HY. Then CR represents the radial acceleration of C toward B. The proof is as follows: We know that the radial component of C's acceleration is

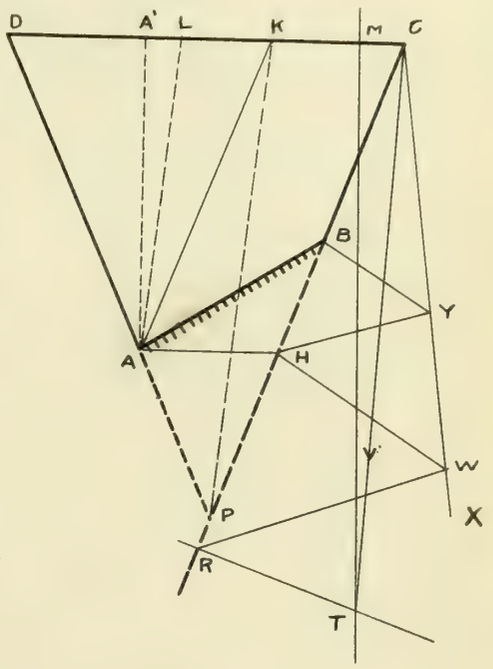


Fig. 29.

$$\frac{v^2}{r} = \frac{\overline{CH}^2}{CB}.$$

The following triangles are similar: CBY and CHW; CHY and CRW; BYH and HWY. We have, therefore,

$$\frac{CR}{CH} = \frac{RW}{HY} = \frac{HW}{BY} = \frac{CH}{CB}.$$

or

$$CR = \frac{\overline{CH}^2}{CB} = \frac{v^2}{r}.$$

Having the radial component CT, we easily find the total acceleration of C to be CT.

Relative Acceleration:— We have already shown that if the vectors representing the accelerations of any two points of a rigid system be laid off from any point as a pole, equal to and parallel to their original lengths, then the closing side of the triangle represents the acceleration of one point relative to the other. Thus in Fig. 30, let A and B have accelerations represented by the vectors AM and BN respectively. From any point O as a pole draw Om and On equal to and parallel to AM and BM respectively. Then mn represents the acceleration of B relative to A, and nm the acceleration of A relative to B. Evidently, the acceleration of A relative to B is equal numerically to the acceleration of B relative to A, but of opposite sense. In all our work, we have assumed that the acceleration of B relative to A is the acceleration of B when the system in which A and B are points is so moved that A has no acceleration. We have, therefore, considered the original system to have a motion of

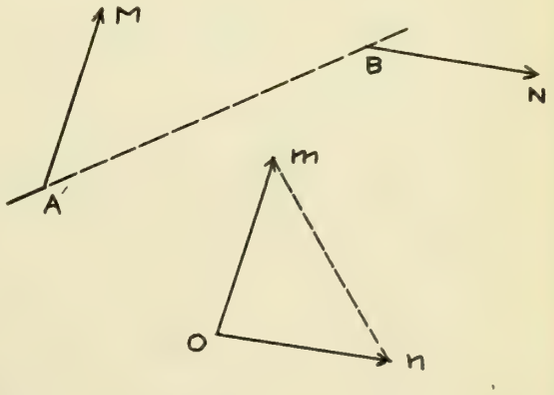


Fig. 30.

rotation plus a translation. In Fig. 30, may consider the rotation as taking place about A as the instant center. From previous considerations we know that the acceleration of any point of a rotating system makes a constant angle with the instantaneous radius of the point. Hence the acceleration of B relative to A makes a fixed angle with the line joining AB. We shall denote this angle by the symbol ϕ. The relative acceleration of any other point C must also make the same angle ϕ with its radius vector.

Acceleration Images. — We shall next consider three points of a rigid system. Let the accelerations of two points, A and B, (Fig. 31.) be known, and represented by the vectors AA' and BB' respectively. Here, again, we consider the motion of the system replaced by a rotation about the point A, and a translation. Our problem is to find the acceleration of the third point C which may be any point of the system.

Choose the point O as a pole and draw Oa and Ob equal to and parallel to AA' and BB' respectively. Then ab represents the acceleration

of B relative to A and makes the angle ϕ with the radius vector to B. Hence, the ends a and b

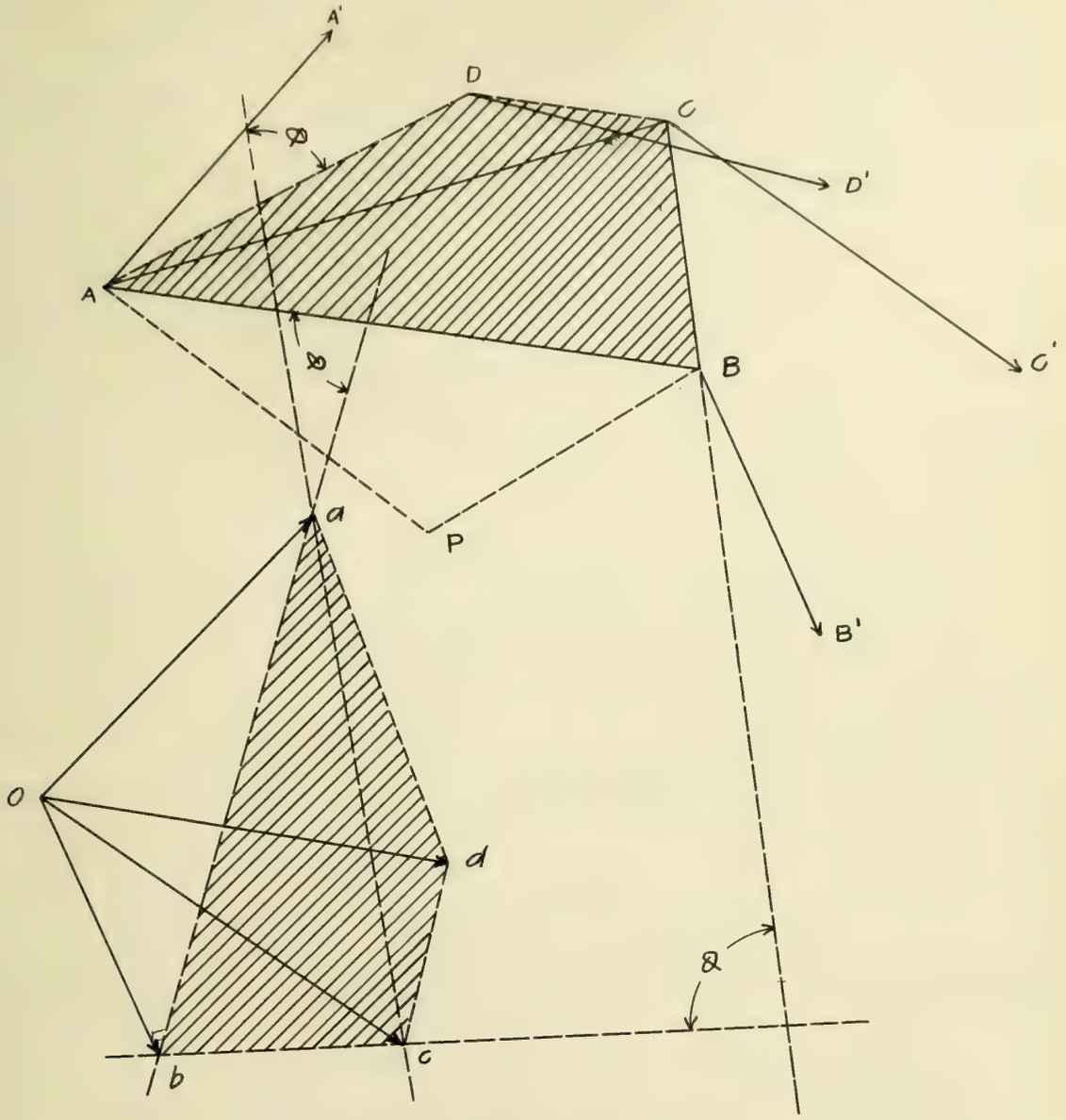


Fig. 31.

of the vectors Oa and Ob lie on a line which makes the angle ϕ with AB. In the same way, we can show that the ends a and c will lie on a line making the same angle ϕ with the line joining A and C. Through a, then, draw a line making the angle ϕ with AC and through

b a line making the same angle ϕ with BC. The intersection of these lines determine c, the end of the vector representing the acceleration of C. By drawing CC' equal to and parallel to OC we have in CC' the vector representing the total acceleration of C.

By construction the sides of the triangle abc make the angle ϕ with the corresponding sides of the triangle ABC; hence the triangles are similar. We have then the following: If from any point as a pole the vectors representing the accelerations of the points of a system are laid off, the ends of these vectors form a system similar to the original system, i.e. an image of the original system.

To find the acceleration of any point D of the system, we locate the corresponding point in the image. Its pole distance is the required acceleration. Thus to find the acceleration of D we locate d in the same position relative to abc that D occupies relative to ABC. Then Od is the required acceleration of D.

Center of Acceleration.— The point O of the image corresponds to the point of the original system that has no acceleration; hence if we

make the triangle ABF similar to abO we shall locate F, the point that is experiencing no change in velocity. This point is called the Center of Acceleration. In case a system has a motion only of rotation then the center of acceleration and center of rotation coincide. In general, however, the two centers are separate and distinct.

It has already been shown that if radii be drawn from the instantaneous center of rotation, to any points of the system, the vectors representing the accelerations of the points are proportional to the distances of the points from the center of rotation and that the vectors all make the same angle with their radii vectors. We shall next show that the same thing is true with respect to the center of acceleration.

In Fig. 32, let the vectors AA' and BB' represent the accelerations of A and B respectively. From any point O lay off these vectors as if to construct an image of the original system. We can find the center of acceleration by constructing triangle QAB similar to Oab, with AB and ab as homologous sides. Draw lines A'Q and B'Q. By construction triangle Oab is similar

to triangle QAB. Therefore

$$\frac{Oa}{QA} = \frac{Ob}{QB}.$$

But $Oa = AA'$ and $Ob = BB'$. Hence

$$\frac{AA'}{QA} = \frac{BB'}{QB}$$

and triangle QBB' is similar to triangle QAA'.

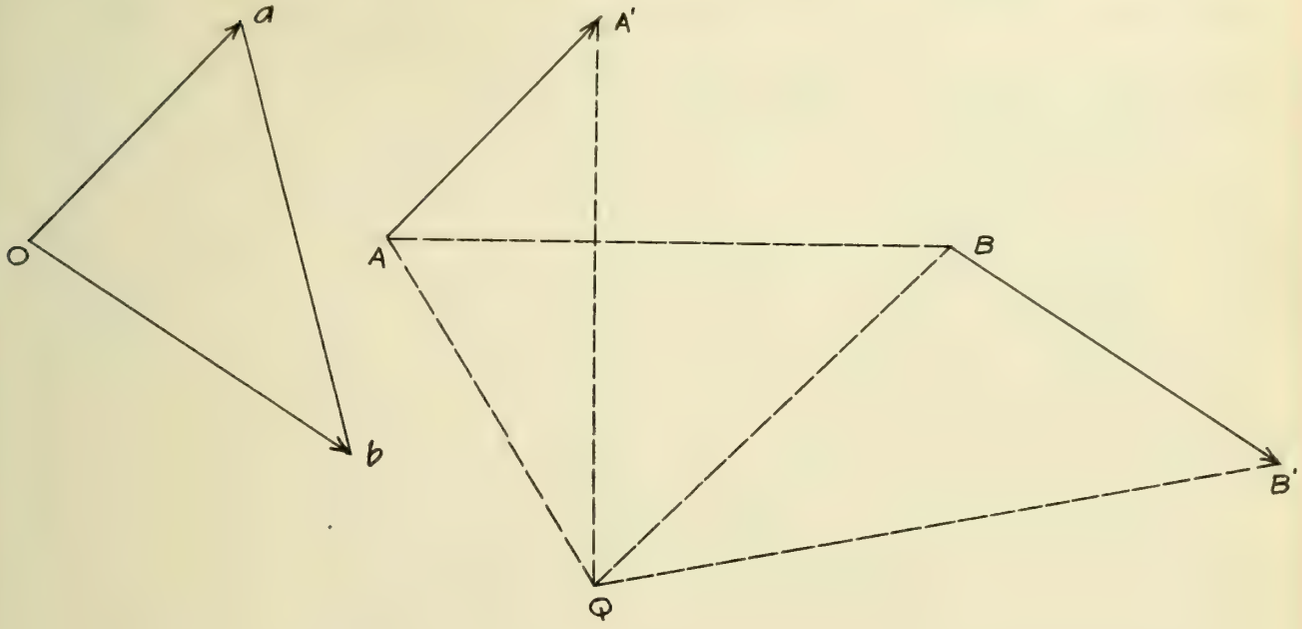


Fig. 32.

It also follows, that angle QBB' is equal to angle QAA'. Therefore, the accelerations of any points of a system are proportional to the length of their radii vectors and make with the radii vectors the same angle.

Atkinson Gas Engine.— In Plate I is shown the skeleton of the Atkinson Gas Engine. The crank AD is assumed to be rotating uniformly. Hence, we have taken its length to represent both the revolved velocity and the acceleration. The velocity polygon is obtained as follows: Draw Oa, from O as a pole, parallel and equal to OA. Draw a line through the pole O parallel to BS, also a line through a parallel to AB. The intersection of these two lines determine the point b. The point E in the image is located by the intersection of the two lines drawn from b and a, parallel to BE and AE respectively. To find f, we merely draw a line through e parallel to EF until it intersects the vertical through O. Then Of is the revolved velocity of the piston F. By repeating this construction, any number of points on the velocity curve V, U, K, M, G, N may be found.

The acceleration of F is found as follows: It will be readily seen that link ABE is the coupler of a four link mechanism. Hence, to find the acceleration of B we proceed as with the four link mechanism. [See Fig. 27.] To find the acceleration of E, draw the acceleration

polygon. To do this choose O' as a pole and draw through O' lines parallel and equal to the vectors representing the accelerations of A and B. These lines in the acceleration polygon are O'A' and O'B' respectively. To locate B' we construct triangle A'B'E' similar to triangle ABE. Then O'B' is the vector representing the acceleration of the point E of link ABE. The method of finding the acceleration of F has already been explained and need not, therefore, be explained here. [See Fig. 33.]

The construction in order to find the center of acceleration for link ABE in the position shown is as follows: Construct triangle MBA similar to triangle O'B'A', with AB and A'B' as homologous sides. Then M is the required center of acceleration. [See Fig. 32]

By repeating the construction outlined above any number of points on the acceleration curve 4, 5, 7, 8, 9, 10, 11, 12, 13 may be found.

Whitworth Quick Return Motion.— In Plate II is shown the skeleton of the Whitworth Quick Return Mechanism. The velocity polygon is constructed in the lower part of the plate. The crank AD is assumed to be rotating uniformly, hence

its length may be taken to represent both the acceleration and the revolved velocity of D. The motion of the block D is a sliding along the rod combined with a rotation about B. Hence by Coriolis law the total acceleration of the pin is the sum of three components.

(1) The acceleration of the point D of link BE.

(2) The acceleration of the block along link BD.

(3) The component $2vw$ perpendicular to BE.

The value of the first component is to be found, for knowing this component we can find the acceleration of any other point of the rod BE as D and hence pass on to the block F and find its acceleration. The component $2vw$ may be found by graphical construction. From a rectangle on the sides BD and BI thus locating the point g. From g draw a line through the middle point P, of segment BD and let it intersect A3 in h. Then Ah is the component $2vw$. The angular velocity w of BD is $\frac{Od}{DB}$, and the velocity of sliding, v, is dd'. Hence

$$2vw = 2 \times dd' \times \frac{Od}{DB}.$$

$$= dd' \times \frac{Od}{PD} \quad \text{since } PD = \tfrac{1}{2} BD.$$

From the similarity of the triangles gAh and gDp we have

$$Ah : D3[= gA] = g\,D[= A3] : PD.$$

or
$$Ah = \frac{D3 \times A3}{PD}$$

But the triangle A3D is equal to the triangle Odd', hence the sides D3 = Od'[mage] and A3 = dd'. Therefore

$$Ah = \frac{Od' \times dd'}{PD}$$
$$= \frac{Od' \times dd'}{\frac{1}{2} BD}$$
$$= 2\,U\,\omega.$$

Here we have given the total acceleration AD, the component $2U\omega = Ah$, and the sum of two components. By a method previously discussed [Fig. 29] find the radial acceleration of D toward B. The end of the vector representing this acceleration falls at 4. Through 4 draw a perpendicular to BE and through h a line parallel to BE. These lines intersect in K. We know that the direction of one of the unknown components must be parallel to BE, also that the end of the vector representing the other component must lie somewhere in the perpendicular K4. We may, therefore, draw the acceleration polygon since we have DA and Ah known both in magnitude and direction. From h draw hK parallel to BE. The line hK inter-

sects 4K in K. The closing side of the polygon is KD and is, therefore, the component due to D's rotation about A.

The acceleration of E is found by constructing triangle BME similar to triangle BKD. Knowing the acceleration of E we can readily find the acceleration of the slide F by methods previously given.

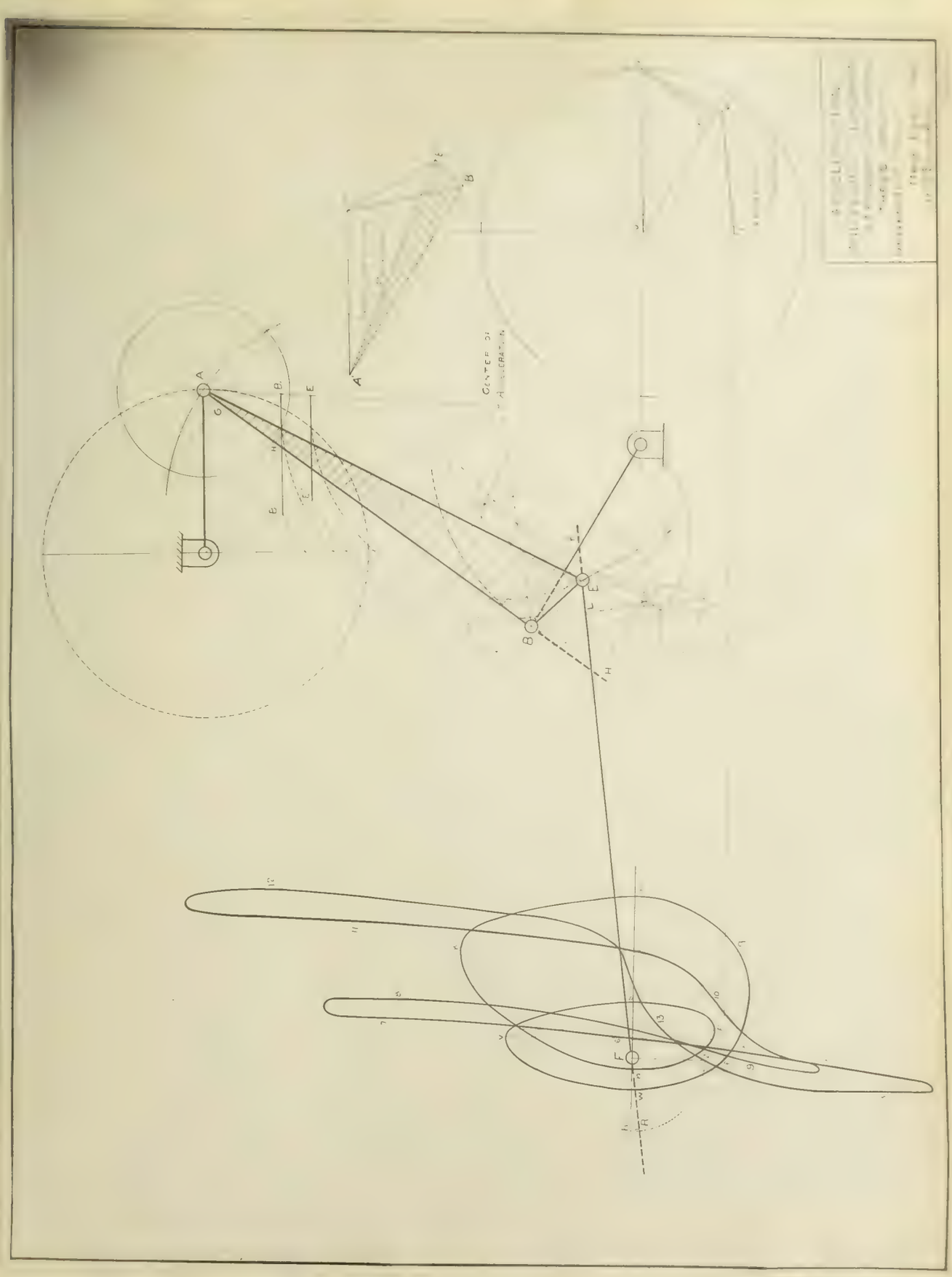
MacCord's Quick Return Motion.— In Plate III is shown a Quick Return Mechanism in which the sliding in the slotted arm of the Whitworth Mechanism is avoided. This is accomplished by connecting two cranks AD and CB of which, AD is the driver, with a short link CD. The method of construction is as follows: Describe about A the circle with radius AD, equal to the length of the driving crank. Divide the circumference of this circle so that the arc XWX shall be to the remainder of the circumference as the time of return is to the time of advance. Bisect chord XX by the perpendicular AB which determines the center B of link BC. The length of the link CD may be varied within certain limits but it must always be greater than WM and less than the diameter of circle WW.D. The

velocity polygon is drawn for the given position of the mechanism. The method of finding the acceleration of E has already been explained and need not, therefore, be repeated here.

German Crank Conveyor. — In Plate IV is the skeleton of a German Crank Drive. OA is the driving link. Here again we have the four link mechanism in which AB is the coupler. At the point D on link BP another link is attached, which is connected by a turning pair to links AE and EF. The construction of the velocity polygon presents no difficulties and will not, therefore, described.

To find the acceleration of the slide F we proceed as follows: The acceleration of B is found by the method already explained in connection with the Four Link Mechanism. Knowing the acceleration of B we easily find the acceleration of D by constructing triangle mDP similar to triangle 1BP. The line Dm represents this acceleration. To find the acceleration of E we make use of the principles explained on pages 48 and 49. The construction is given below. The projection of D's acceleration on DE is Dm. Employing the method of construction

illustrated in Fig. 29, we find the radial component of D's acceleration toward E to be DS. The difference between the projection of D's acceleration and the projection of D's acceleration relative to E is Sm. Laying off this distance from E we have in En the projection of E's total acceleration on DE. By the same method we find the projection of E's acceleration on EA to be Ey. Therefore the intersection of the perpendiculars to ED and EA through n and y respectively, locates the end of the vector representing the total acceleration of E. These lines intersect in q and Eq represents the total acceleration of E. To find the acceleration of F we proceed in a manner already described on page 48.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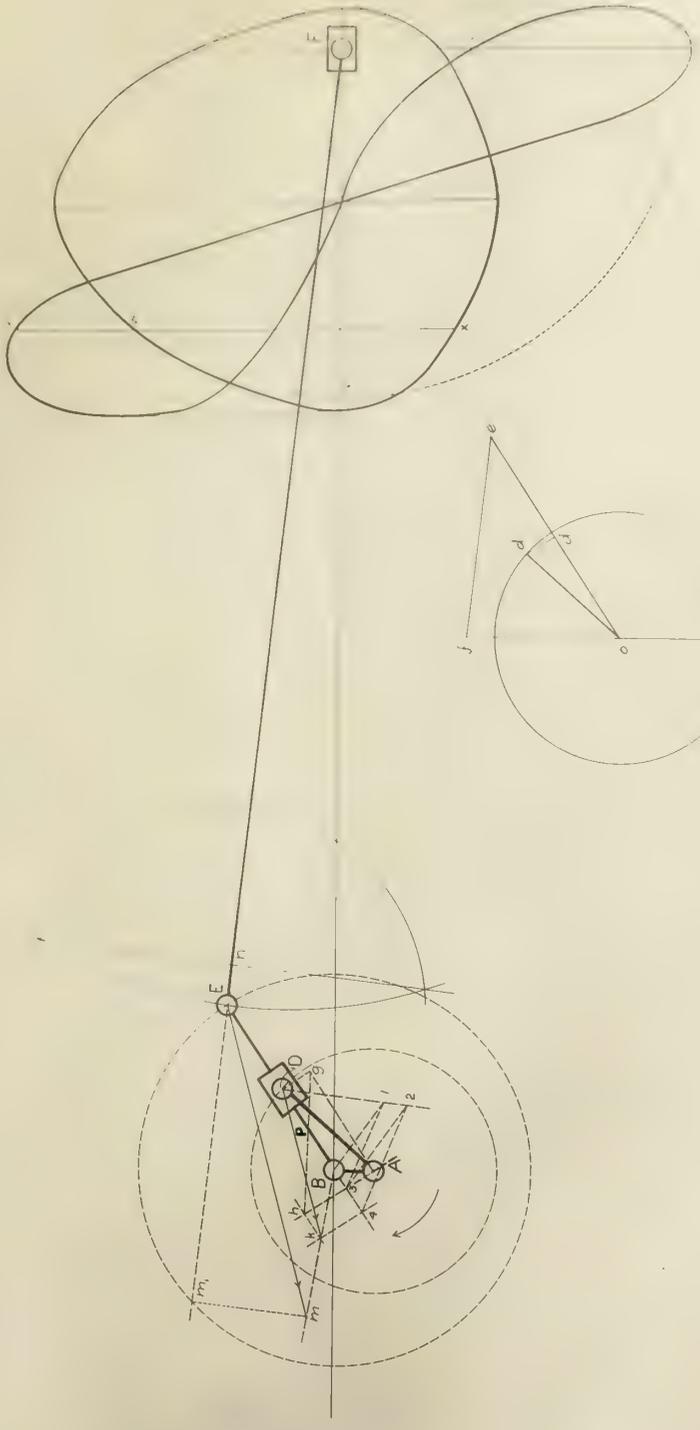
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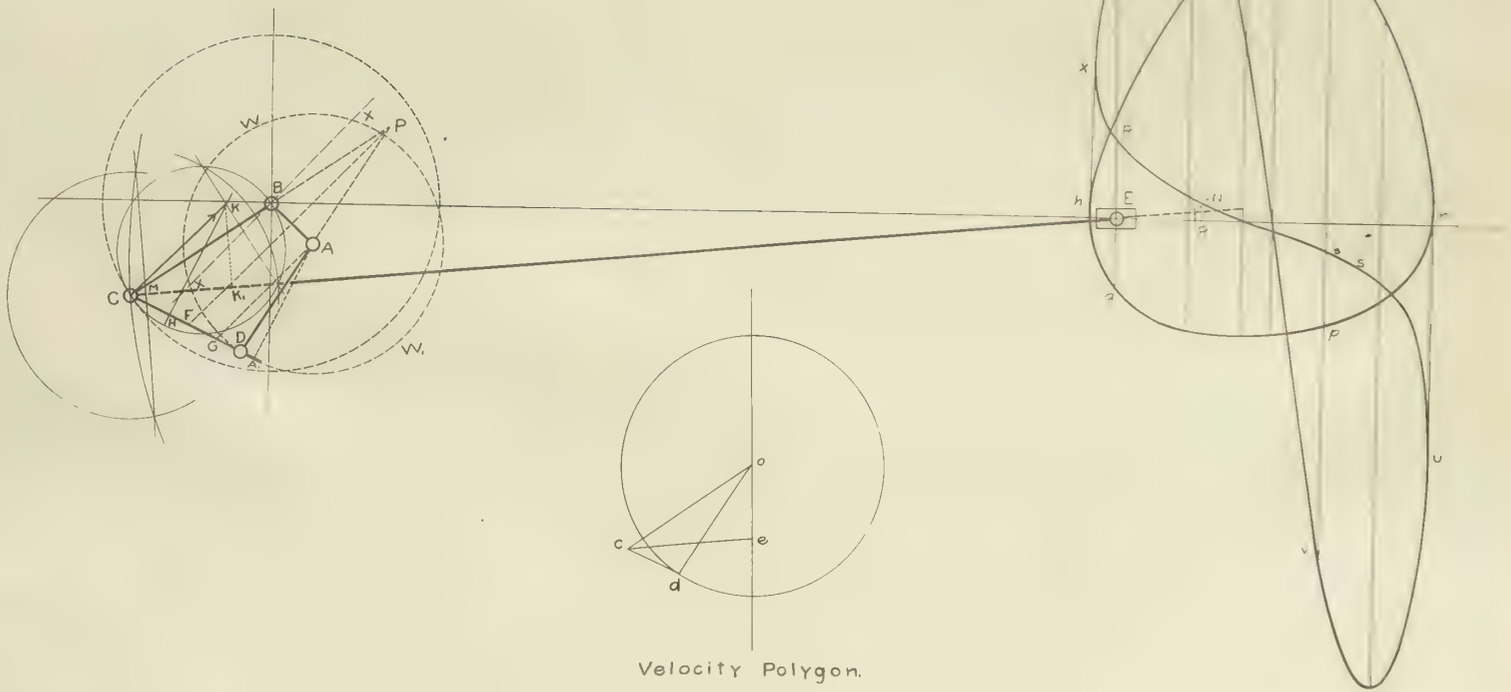
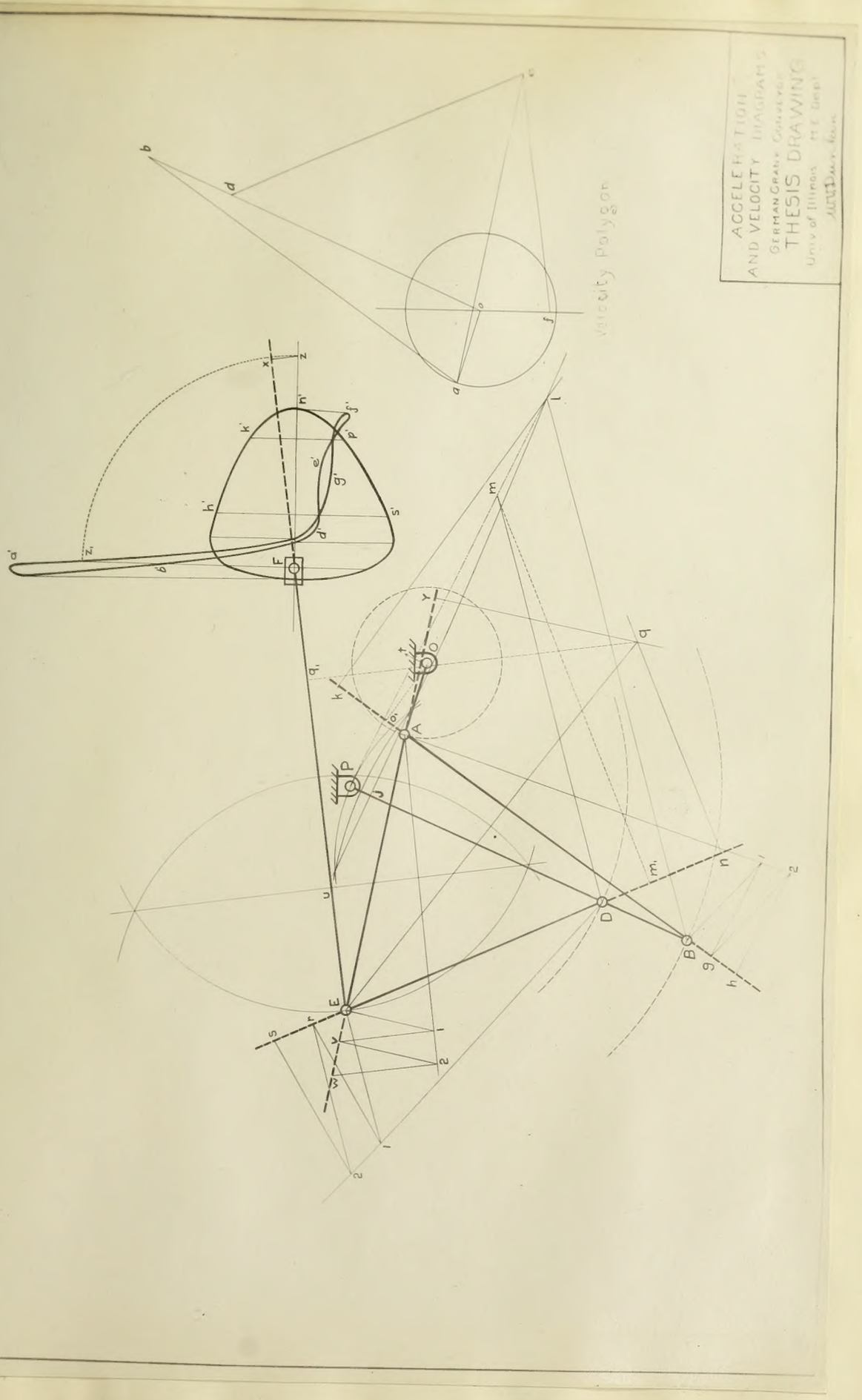


Velocity Polygon.

ACCELERATION
AND VELOCITY DIAGRAMS
THESIS DRAWING

11


ACCELERATION
AND VELOCITY DIAGRAMS
GERMAN GRADE CONSTRUCTION
THESIS DRAWING
Univ of Illinois M.E. Dept
Velocity Polygon

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